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PROJECT TUBEFLIGHT

THE DETERMINATION OF TUBEFLIGHT
VEHICLE STABILITY DERIVATIVES BY
AN ANALOG REGRESSION TECHNIQUE

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SUMMARY

The technique of analog regression is applied to the measured motion of a fluid-supported vehicle traveling in a tube for the purpose of finding the vehicle stability derivatives.

The model chosen to illustrate the application of this technique is a hypothetical full-scale design representative of a typical Tubeflight vehicle. Assuming linear equations with constant coefficients, accelerations in the vehicle plane of symmetry after certain types of disturbance are determined by Laplace Transform methods, plotted, and programmed into an analog computer by means of a Curve Follower similar to the manner in which flight test data could be processed. The analog computer program used to demonstrate the analog regression technique is a first order correlation program, with corrections derived to give improved "second order" coefficients. More sophisticated programs are presented but not tested, as the number of analog amplifiers required rises rapidly with the number of stability derivatives to be determined.

The results indicate the feasibility of the technique.

SYMBOLS

a	center of gravity location, $\frac{x_{CG}}{l} - \frac{1}{2}$
a_1, a_2	s coefficient of fourth order factors
A_i	first numerator multiplier, for i^{th} power in numerator
b	intercept of straight line fit
b_1, b_2	unit coefficient of fourth order factors
B	coefficient of s^3 term in fourth order polynomial
B_i	second numerator multiplier, for i^{th} power in numerator
c	airfoil chord
C	coefficient of s^2 term in fourth order polynomial
C_i	third numerator multiplier, for i^{th} power in numerator
C_D	vehicle drag coefficient, $\frac{D}{\rho S_m}$
C_{d_p}	zero-lift pad section drag coefficient, $\frac{d_p}{\rho_t c}$
C_j	jet flap momentum coefficient, $\frac{\rho_j V_j^2 \delta}{\rho_t c}$
C_L	vehicle lift coefficient, $\frac{L}{\rho S_m}$
C_M	pitching moment coefficient, $\frac{M}{\rho S_m l}$
C_T	thrust coefficient, $\frac{T}{\rho S_m}$
D	coefficient of s^1 term in fourth order polynomial
D	drag of vehicle (lbs)
e	input to non-linear control system
E	coefficient of s^0 term in fourth order polynomial
g	acceleration of gravity, 32.2 ft/sec^2
G	integral of square error
h	airfoil ground clearance
I_b	nondimensional pitch inertia, $\frac{2 I}{\rho S_m l^4}$
I	vehicle pitch inertia
j	$\sqrt{-1}$
k_j	numerators of partial fractions expansion

L	vehicle lift
m'	slope of straight line fit
m	vehicle mass
M	pitching moment (lb. ft.)
M	Mach number
N()	a nonlinear function
N	the describing function
N _r	real component of N
N _i	imaginary component of N
p(s)	a polynomial in s
P _j	the poles of $\frac{p(s)}{f(s)}$, the zeros of $f(s)$
q(s)	a polynomial in s of higher order than p(s)
q _p	passageway dynamic pressure
q	freestream dynamic pressure
s	complex frequency
S _m	maximum vehicle cross section area
S _T	tube cross sectional area
t	time (sec.)
T	thrust (lb.)
u(t)	disturbance velocity along vehicle longitudinal axis
U(t)	velocity along vehicle longitudinal axis
U(s)	Laplace transform of disturbance velocity
x _{CG}	position of vehicle center of gravity (from nose)
z(t)	heave disturbance
Z	heave position

$z(s)$ Laplace transform of heave disturbance

z_j the zeros of $P(s)$

Subscripts

$()_0$ initial steady state value

Greek Symbols

α angle of attack

α' second order system decay rate, $\frac{1}{2} z_{\dot{y}}$

β second order system damped frequency $\sqrt{z_{\gamma} - \frac{1}{4} z_{\dot{y}}^2}$

γ vehicle flight path angle

δ jet thickness

ϵ error, $\ddot{z}_0 - \sum_i z_{x_i} x_i$

ϵ airfoil lower surface inclination to ground plane

$\vartheta(t)$ pitch angle disturbance

$\theta(s)$ Laplace transform of pitch angle

θ pitch angle

ζ second order system damping ratio

μ nondimensional mass, $\frac{2 m}{\rho S_m l}$

ρ air density (0.00238 slug/ft²)

ω angular frequency

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I. INTRODUCTION

The development of the concept of a self-propelled, aerodynamically supported vehicle capable of high speed travel in a tube will soon reach the point at which instrumented flight tests of small scale test models will be conducted. Some of the important results of such tests will be performance characteristics and stability data (vehicle motions vs time) which can be used to design more advanced scale models and eventually large commercial vehicles. The stability data desired will be in terms of vehicle stability derivatives or system transfer function coefficients. In either case a method of data reduction must be chosen which will give accurate reproducible results.

The analog regression technique of analysis in the time domain can be used as a method of stability derivative extraction from a knowledge of the trajectory.

Regression is the branch of statistics in which relations between two or more populations representing different variables are found. The method used to find such relations is termed regression analysis. The populations may be either finite (discrete data) or infinite (continuous data). A typical result for discrete data is the familiar fit of a straight line to a set of N points, (X_i, Y_i) . If $Y = mX + b$ is the assumed form of dependence,

$$\sum_{i=1}^N Y_i = m \sum_{i=1}^N X_i + Nb$$

$$\sum_{i=1}^N X_i Y_i = m \sum_{i=1}^N X_i^2 + b \sum_{i=1}^N X_i$$

are the two simultaneous equations for m and b for a least-error-squared fit. A fit to continuous data can be made by letting N grow toward infinity while

$N \Delta t = t$ is held constant. Then

$$\int_0^t Y dt = m \int_0^t X dt + bt$$

$$\int_0^t XY dt = m \int_0^t X^2 dt + b \int_0^t X dt$$

are the simultaneous equations whose solutions are m and b for a straight line having the least error squared (integrated over time) (Ref. 1). The method of steepest descents outlined in the body of this thesis is a powerful regression technique which gives the same results as the classical solution for a straight line fit.

Other computer techniques (Ref. 2) can produce more accurate results. Digital computers require discrete numerical data so that for good results input data points must be closely spaced in time and have at least as many significant figures as is required of the output.

The analog computer's advantages are its ability to handle continuous data in analog form (sensor outputs may be voltages used directly as inputs) and in "real time". It is possible that vehicle modification during testing could be based on just-obtained derivative information, resulting in a rapid approach to a satisfactory vehicle configuration. The use of the regression technique in an attempt to determine the stability and control derivatives of an aircraft in the lateral mode of motion is discussed. The point is made in this reference that instrumentation must be highly accurate, without time lags between measured quantities. It is even stated that instrumental accuracy is more critical than the data reduction method. For the remainder of this paper it will be assumed that perfect data is available to the computer, independent of the number of channels of information. Rubin (Ref. 1)

derives the regression technique very clearly and includes simple analog computer diagrams.

The analyses presented in the following pages are based upon linearized equations of motion and assumes that the vehicle is a rigid body with no inherent angular momentum. Revolving machinery on the vehicle could necessitate a preliminary data reduction to account for the gyrodynamic effects produced. A flexible vehicle is difficult to analyze, but the assumption of rigidity is made only in order that the equations of motion may be simplified to facilitate their solution.

The major problem is perhaps the fact that, while the stability derivatives are assumed to be constant, they have been shown by theoretical analysis and experiment to vary considerably for relatively small displacements. The results of computer analysis of the motion following a finite disturbance will necessarily be rather a best average of values over the conditions assumed by the vehicle during the data run than a value of the derivative for a particular flight condition. For such problems the solution can only be a limit as the disturbance becomes infinitesimal. Averaged values of stability derivatives obtained from data runs including larger disturbances are of use, of course.

In the study of non-linear systems it is useful to introduce a "describing function" or amplitude-dependent transfer function approximating a non-linear function. The best linear fit to data at different amplitudes of vehicle motion establishes each derivative as a describing function, analysis of which can even determine the derivative as a nonlinear function of vehicle conditions (Appendix 1).

An alternate way of handling non-linear dependence is to assume the form of dependence. Powers other than unity, sine and co-sine functions, ex-

ponentials and many other non-linear forms of dependence can be programmed or approximated. The program will correlate to find the coefficient which best fits the assumed form of dependence to the input data. A certain amount of judgement and experience would be required to make use of this technique.

The least squares fit assuming linear dependence is likely to give good qualitative and acceptable quantitative results, and serves well as an illustration of the technique. The analysis in this paper will deal with a tubeflight vehicle having three degrees of freedom in the longitudinal mode, with each degree of freedom represented by a linear equation in the state variables. The problem is to determine the coefficients multiplying each variable in the equations from the trajectory.

Flight test data is at present unavailable, but can be "generated" by Laplace transform techniques, or by programming an analog computer with the equations of motion (a simulator program). The coefficients of the terms in the equations and initial conditions must be assumed in either case. The data generated can be put into a derivative-finding program, which must determine the assumed coefficients with reasonable accuracy. Because of the accuracy of the transform technique, it is used here. As a check the simulator program output may be compared with the results.

It would be no more difficult to work in the frequency domain and to find transfer function coefficients, but as the analysis would no longer be in the time domain, a new continuous variable would have to be used in place of time. This practically precludes "real time" solution. The variables (frequency, amplitude, and phase angle) would most likely be recorded as discrete data points, making the digital computer the natural tool to apply to this problem.

When a number of the stability derivatives can be estimated a priori, this information can be used in various ways to influence the computer output. The least squares program can be simplified when such a priori information is set into the computer as constants rather than as variables to be determined. Other programs (Ref. 2) use a priori data as though it were input data so that the fit error includes departures from a priori values of the derivatives.

II. ANALYSIS

A. The Equations of Motion of the Vehicle

The motion of vehicle can be represented in any of a number of reference frames, but for ease of solution for an expression of vehicle stability derivatives, the so-called "stability axes" will be employed. The three orthogonal axes are simply those of the principal moments of inertia. Figure 1 illustrates the approximate geometry of a tubeflight vehicle and the significance of the variables of longitudinal motion.

The three independent variables (degrees of freedom) are chosen to be U, velocity along the longitudinal axis, Z, position on the vertical axis, and θ , pitch angle, each measured from the tube centerline to the vehicle. Vehicle instrumentation, such as pad proximity sensors, can be set up easily to read out the required variables.

A glance at Figure 1 will verify that the longitudinal dynamics of the vehicle are expressed by

$$\begin{aligned}
 m\dot{U} &= T - D \cos \alpha + L \sin \alpha - mg \sin \theta \\
 m\ddot{Z} &= L \cos \alpha - mg \cos \theta + D \sin \alpha \\
 I\ddot{\theta} &= M(Z, \dot{Z}, \theta, \dot{\theta})
 \end{aligned}$$

Assuming small angles for the purpose of linearization of the equations of motion,

$$\begin{array}{lll}
 \cos \gamma \cong 1 & \cos \alpha \cong 1 & \cos \theta \cong 1 \\
 \sin \gamma \cong \gamma & \sin \alpha \cong \alpha & \sin \theta \cong \theta
 \end{array}$$

and $\gamma + \alpha = \theta$, $\gamma = \tan^{-1}(\dot{Z}/V)$

thus

$$\theta = \alpha + \dot{z}/V$$

In steady flight,

$$T = D \cos \alpha - L \sin \alpha$$

$$T \approx D - L \alpha$$

and

$$L = mg \cos \gamma - T \sin \alpha$$

$$L = mg - T \alpha$$

Considering small disturbances from equilibrium steady-state flight

$$z - z_0 = \gamma, \quad \dot{z} - \dot{z}_0 = \dot{\gamma} = d\gamma/dt \quad \text{if } \dot{z}_0 = 0$$

$$\theta - \theta_0 = \alpha, \quad \dot{\theta} - \dot{\theta}_0 = \dot{\alpha} = d\alpha/dt \quad \text{if } \dot{\theta}_0 = 0$$

$$U - U_0 = u$$

the linearized non-steady flight equations are

$$m\dot{U} \approx T - D + L\alpha - mg\theta$$

$$\approx T - D - mg\gamma$$

$$\approx T - D - \frac{mg}{V}\dot{z}$$

$$m\ddot{z} \approx L - mg + D\alpha$$

$$\approx L - mg + D\theta - D\frac{\dot{z}}{V}$$

$$I\ddot{\theta} \approx \frac{dM}{dz}(z - z_0) + \frac{dM}{d\dot{z}}(\dot{z} - \dot{z}_0)$$

$$+ \frac{dM}{d\theta}(\theta - \theta_0) + \frac{dM}{d\dot{\theta}}(\dot{\theta} - \dot{\theta}_0)$$

The equations of motion are nondimensionalized as follows:

$$\eta = \frac{y}{l}$$

$$\dot{\eta} = \frac{\dot{y} l}{l V} = \frac{\dot{y}}{V}$$

$$z = z$$

$$\dot{z} = \frac{\dot{z} l}{V}$$

$$u' = \frac{u}{V}$$

$$\mu = \frac{m}{\frac{1}{2} \rho S_m l} ,$$

$$I_b = \frac{I}{\frac{1}{2} \rho S_m l^3}$$

$$C_D = \frac{D}{\frac{1}{2} \rho S_m} ,$$

$$C_T = \frac{T}{\frac{1}{2} \rho V^2 S_m}$$

$$C_L = \frac{L}{\frac{1}{2} \rho V^2 S_m} ,$$

$$C_j = \frac{\rho_j V_j^2 \delta}{\rho_T c}$$

Finally, the non-steady equations of motion can be written as

$$\begin{aligned} \mu \dot{u} &= C_T - C_D - \mu \dot{\eta} \left(\frac{l g}{V^2} \right) \\ &= C_{x_u} u + C_{x_\eta} \eta + C_{x_{\dot{\eta}}} \dot{\eta} + C_{x_z} z \end{aligned}$$

$$\mu \ddot{\eta} = C_{z_\eta} \eta + C_{z_{\dot{\eta}}} \dot{\eta} + C_{z_z} z + C_{z_{\dot{z}}} \dot{z} + C_{z_u} u$$

$$I_b \ddot{z} = C_{M_\eta} \eta + C_{M_{\dot{\eta}}} \dot{\eta} + C_{M_z} z + C_{M_{\dot{z}}} \dot{z}$$

Chord line angle to pad lower surface 1.72°

Lower surface inclination ξ ----- 0.03 radians (1.7°)

Other Characteristics

$$C_{d_p} \text{ ----- } 0.006$$

Vertical fin at rear for later control and stability.

Cruise speed ----- 250 kts. = 420 fps

Jet flaps in operation: $C_j = 0.083$ each pad

Transfer passage dynamic pressure ratio

Transfer passage Mach number $M_o = 0.76$

$$\mu = 271$$

$$l_b = 22.6$$

Cooke evaluated the stability derivative to be

$$C_{z_\eta} = 35.0 - \mu \left(\frac{313.5 \text{ fps}}{V} \right)^2 + \frac{\rho_p}{\rho_\infty} (25.0 - 1539 \xi)$$

$$C_{z_j} = -58.80$$

$$C_{z_\beta} = \left[35.0 + \frac{\rho_p}{\rho_\infty} (25.0 - 1539 \xi) \right] a + \frac{\rho_p}{\rho_\infty} (18.81 - 4.51 C_{d_p})$$

$$C_{z_\delta} = -58.80 a + 5.35$$

$$C_{M_\eta} = \left[35.0 + \frac{\rho_p}{\rho_\infty} (25.0 - 1539 \xi) \right] a - 4.51 \frac{\rho_p}{\rho_\infty} C_{d_p}$$

$$C_{M_j} = -5.35 - 58.80 a$$

$$C_{M_\beta} = C_{z_\eta} (a^2 + 0.08166 + 18.81 \frac{\rho_p}{\rho_\infty} a)$$

$$C_{M_\delta} = -58.80 a^2 - 4.81$$

Forces due to velocity are approximately represented by

$$C_{x_u} = -3 C_{T_o} - M_o \left. \frac{dC_d}{dM} \right|_o, \quad C_{z_u} = M_o \left. \frac{dC_L}{dM} \right|_o$$

Wind tunnel tests of an elliptical section with nearly flat lower surface 0(10%) thick, conducted at R.P.I. by Duffy (Ref. 6), at

$$C_j \cong 0.083 \quad \frac{h}{c} \cong 0.10 \quad \alpha \cong 0^\circ \text{ to } 5^\circ$$

yield

$$C_{x_\eta} \cong -0.36 \quad C_{x_\alpha} \cong -0.004 \frac{\text{deg}}{\text{deg}} (\alpha = \vartheta - \frac{\dot{\gamma}}{U})$$

Assuming the Prandtl Glauert compressibility scaling law or pressure coefficients,

$$C_L = \frac{C_L|_{N=0}}{\sqrt{1-M^2}}$$

$$C_D = \frac{C_D|_{u=0}}{\sqrt{1-M^2}}$$

drag is estimated at 7,550 lbs based on $C_D|_{u=0}$

$$\rho S_m = \frac{1}{2} \rho V^2 S_m = \frac{1}{2} (0.00238 \frac{\text{slug}}{\text{ft}^3}) (420 \frac{\text{ft}}{\text{sec}})^2 (63.6 \text{ ft}^2) = 13,300 \text{ lbs.}$$

$$C_D|_{u=0} = \frac{7,750 \#}{\rho S_m} = 0.57 = C_T|_{u=0}$$

$$C_L|_{u=0} = \frac{90,000}{\rho S_m} = 6.77$$

$$C_{x_u} = C_D|_{u=0} \left(-3 - M_0 \frac{d}{dM} \left(\frac{1}{\sqrt{1-M^2}} \right) \right) = 0.57 \left(-3 - \frac{M_0^2}{(\sqrt{1-M_0^2})^3} \right)$$

$$= -0.57 \left(3 + \frac{(0.76)^2}{(\sqrt{1-(0.76)^2})^3} \right) = -2.9$$

Similarly,

$$C_{z_u} = 6.77 \left(\frac{(0.76)^2}{(\sqrt{1-(0.76)^2})^3} \right) = 14.2$$

Evaluated

$$C_{z_\eta} = -180$$

$$C_{z_{\dot{\eta}}} = -58.8$$

$$C_{z\alpha} = 58.2$$

$$C_{z\dot{\alpha}} = 7.23$$

$$C_{M\eta} = 0.85$$

$$C_{M\dot{\eta}} = -3.47$$

$$C_{M\alpha} = -8.82$$

$$C_{M\dot{\alpha}} = -4.87$$

where α is measured in radians. The nondimensionalized equations are thus

$$\begin{aligned} 271 \dot{u}' &= -2.9 u' - 0.36 \eta - (271 \frac{lg}{V^2}) \dot{\eta} - 0.004 /_{\text{deg}} (\alpha - \frac{\dot{z}}{V}) \\ &= -2.9 u' - 0.36 \eta - 6.55 \dot{\eta} - 0.004 \alpha /_{\text{deg}} \end{aligned}$$

$$\begin{aligned} 271 \ddot{\eta} &= -180 \eta - 58.8 \dot{\eta} + 58.2 \alpha /_{\text{rad}} + 7.23 \dot{\alpha} /_{\text{rad}} + 14.2 u' \\ &= -180 \eta - 58.8 \dot{\eta} + 1.02 \alpha /_{\text{deg}} + 0.126 \dot{\alpha} /_{\text{deg}} + 14.2 u' \end{aligned}$$

$$22.6 \ddot{\alpha} /_{\text{rad}} = 0.85 \eta - 3.47 \dot{\eta} - 8.82 \alpha /_{\text{rad}} - 4.87 \dot{\alpha} /_{\text{rad}}$$

or

$$0.395 \ddot{\alpha} /_{\text{deg}} = 0.85 \eta - 3.47 \dot{\eta} - 0.154 \alpha /_{\text{deg}} - 0.085 \dot{\alpha} /_{\text{deg}}$$

When working with flight data it is perhaps more convenient to gather all information in dimensional form and to nondimensionalize later. For $l = 137$ ft, $\rho S_m = 13,300\#$,

$$V = 420 \text{ fps ,}$$

the equations in dimensional form are

$$2,790 \dot{u} = -92 u - 35 \gamma - 207 \dot{\gamma} - 53.3 \vartheta_{\text{deg}}$$

$$2,790 \ddot{\gamma} = -17,500 \gamma - 1,860 \dot{\gamma} + 13,500 \vartheta_{\text{deg}} + 550 \dot{\vartheta}_{\text{deg}} \\ + 450 u$$

$$76,400 \ddot{\vartheta}_{\text{deg}} = 11,300 \gamma - 15,000 \dot{\gamma} - 280,000 \vartheta_{\text{deg}} \\ - 50,500 \dot{\vartheta}_{\text{deg}}$$

Normalized with respect to the coefficients of the highest order derivatives,

$$\dot{u} = -0.033 u - 0.125 \gamma - 0.074 \dot{\gamma} - 0.019 \vartheta_{\text{deg}}$$

$$\ddot{\gamma} = -6.27 \gamma - 0.667 \dot{\gamma} + 4.84 \vartheta_{\text{deg}} + 0.197 \dot{\vartheta}_{\text{deg}} \\ + 0.161 u$$

$$\ddot{\vartheta}_{\text{deg}} = 0.148 \gamma - 0.197 \dot{\gamma} - 3.67 \vartheta_{\text{deg}} - 0.66 \dot{\vartheta}_{\text{deg}}$$

B. Prediction of Vehicle Motion After a Disturbance

Laplace transforms allow the simplest method of solution of this system of equations. Block diagram or signal flow graph techniques (as developed and discussed in Reference 4) may be applied to this linear, time-invariant set of equations.

The transformed equations are

$$(S + 0.033) U(s) = (-0.074 S - 0.125) \ddot{z}(s) - 0.019 \dot{z}(s)$$

$$(S^2 + 0.667 S + 6.27) Z(s) = (0.197 S + 4.84) \dot{z}(s) + 0.161 U(s)$$

$$(S^2 + 0.66 S + 3.67) \dot{z}(s) = (-0.197 S + 0.148) Z(s)$$

or

$$U(s) = \frac{-0.074 S - 0.125}{S + 0.033} Z(s) + \frac{-0.019}{S + 0.033} \dot{z}(s)$$

$$Z(s) = \frac{0.197 S + 4.84}{S^2 + 0.667 S + 6.27} \dot{z}(s) + \frac{0.161}{S^2 + 0.667 S + 6.27} U(s)$$

$$\dot{z}(s) = \frac{-0.197 S + 0.148}{S^2 + 0.66 S + 3.67} Z(s)$$

These equations can be visualized in block diagram form, as in Figure 2.

The system characteristic equation, as determined by Mason's loop rule (Ref. 4, pp. 48-49) is

$$S^5 + 1.36 S^4 + 10.47 S^3 + 7.88 S^2 + 23.3 S + 0.81 = 0$$

The system is stable, according to Routh's Criterion (Ref. 4, p. 113 ff.).

The Routh array appears as follows:

S^5	1	10.47	23.3	0
S^4	1.36	7.88	0.81	0
S^3	4.67	22.1	0	
S^2	1.44	0.81	0	
S^1	19.5	0		
S^0	0.81			

The first column is composed of positive numbers only, the criterion for stability.

This fifth order equation has one negative real root and two pairs of complex conjugate roots, closely associated with those of the uncoupled systems ($\dot{u} = -0.033u$, $\ddot{y} = -6.27y - 0.67\dot{y}$, $\ddot{x} = -3.67x - 0.66\dot{x}$). The real root is very nearly the solution of $23.3S + 0.81 = 0$, or $S = -0.0348$. Using the standard technique to find the real root of this polynomial (Newton-Raphson), the root is found to be $S = -0.0350$. The fourth order equation resulting when $S + 0.0350$ is divided into the fifth order characteristic equation above is

$$S^4 + 1.325 S^3 + 10.424 S^2 + 7.415 S + 23.04 = 0$$

One standard technique used to determine the roots of such a fourth order polynomial is the graphical method innovated by Zimmerman in the 1930's (Ref. 7). The fourth order polynomial is broken into two second order poly-

nomials:

$$S^4 + BS^3 + CS^2 + DS + E = (S^2 + a_1S + b_1)(S^2 + a_2S + b_2)$$

Thus

$$B = a_1 + a_2$$

$$C = a_1a_2 + b_1 + b_2$$

$$D = a_1b_2 + b_1a_2$$

$$E = b_1b_2$$

Thus

$$C = b_1 + \frac{E}{b_1} + a_1(B - a_1)$$

or

$$a_1 = \frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 - \left(C - b_1 - \frac{E}{b_1}\right)}$$

also

$$D = a_1 \frac{E}{b_1} + (B - a_1)b_1$$

or

$$a_1 = \frac{D - Bb_1}{\frac{E}{b_1} - b_1}$$

Plots of the two functions $a_1(b_1)$ yield two curves, with two intersections, where both equations are satisfied simultaneously. These points are (a_1, b_1) and (a_2, b_2) , the unknowns in the factored expression.

The plots can be made relatively accurate by considering only the region in the (a, b) plane near $(0.667, 6.27)$ and $(0.66, 3.67)$, since

the uncoupled heave and pitch characteristic equations are

$$S^2 + 0.667S + 6.27 = 0$$

and

$$S^2 + 0.66S + 3.67 = 0$$

respectively. The graphical method (Fig. 8) yielded the first approximation

$$a_1 = 0.40, \quad b_1 = 6.47$$

$$\left(\text{and } a_2 = B - a_1 = 0.925, \quad b_2 = \frac{E}{b_1} = 3.58 \right)$$

which was improved by division of

$$S^2 + (0.40 + \delta)S + (6.47 + \epsilon)$$

into

$$S^4 + 1.325S^3 + 10.24S^2 + 7.415S + 23.04$$

The result was

$$S^2 + (0.925 - \delta)S + (3.58 - \epsilon - 0.525\delta)$$

and a remainder

$$(-0.01 + 3.1\delta - 0.525\epsilon)S + (-0.16 + 3.4\delta + 2.9\epsilon)$$

which was set to zero, determining δ and ϵ as 0.011 and 0.04 respectively.

The two second order factors of the fourth order polynomial are thus

$$S^2 + 0.411S + 6.51$$

and

$$S^2 + 0.914S + 3.54$$

(see Appendix 2)

This ability to factor the characteristic polynomial is important because it forms the denominator of every Laplace transform of the vehicle's motions. The partial fraction techniques (used to evaluate the solutions in terms easily inverted to the time domain) depends upon the factors of the denominator being known.

The Residue method of solving partial fractions (Ref. 4, p. 27) states that a ratio of polynomials

$$\frac{P(s)}{Q(s)} = \frac{(S-Z_1)(S-Z_2)\cdots(S-Z_m)}{(S-p_1)(S-p_2)\cdots(S-p_n)} \quad n > m$$

may be split into its partial fractions

$$\frac{P(s)}{Q(s)} = \frac{K_1}{S-p_1} + \frac{K_2}{S-p_2} + \cdots + \frac{K_n}{S-p_n}$$

For distance p_i ,

$$K_i = \left. \frac{(S-p_i)P(s)}{Q(s)} \right|_{S=p_i}$$

Here the five p_i are

$$\begin{aligned} & -0.035 \\ & -0.205 \pm j2.54 \\ & -0.457 \pm j1.82 \end{aligned}$$

where the latter are the complex pairs found by factoring the second order factors (application of the quadratic formula).

Each fraction numerator of the partial fraction expansion depends linearly on the numerator in the ratio of polynomials to be expanded.

This polynomial is a sum of powers of S , each multiplied by a co-efficient. The coefficients will vary depending on the disturbance to which the system has been subjected and the particular variable under observation. Rather than perform the partial fractions expansion after determining each transform (ratio of polynomials in S) an expansion for each different power of S to be expected in the numerator will be found beforehand. Then only multiplication by the coefficients in the numerator and summation of the results is required to give the partial fractions expansion. Let

$$\frac{S^i}{(S+0.035)(S+0.205+j2.54)(S+0.205-j2.54)(S+0.457+j1.82)(S+0.457-j1.82)}$$

$$= \frac{A_i}{S+0.035} + \frac{B_i}{S+0.205+j2.54} + \frac{B_i^*}{S+0.205-j2.54} + \frac{C_i}{S+0.457+j1.82} + \frac{C_i^*}{S+0.457-j1.82}$$

$$= \frac{A_i}{S+0.035} + 2 \operatorname{Re} \left\{ \frac{B_i}{S+0.205+j2.54} \right\} + 2 \operatorname{Re} \left\{ \frac{C_i}{S+0.457+j1.82} \right\}$$

where $()^*$ denotes complex conjugate: $(a + jb)^* = (a - jb)$, and $\operatorname{Re} \{ \}$ denotes the real part of the complex number $\{ \}$. The A_i , B_i , and C_i are found by the residue method to be

$$A_i = \frac{S^i}{(S+0.205+j2.54)(S+0.405-j2.54)(S+0.457+j1.82)(S+0.457-j1.82)} \Big|_{S=-0.035}$$

or

$$A_i = 0.044 (-0.035)^i$$

$$B_i = \frac{S^i}{(S+0.035)(S+0.205-j2.54)(S+0.457+j1.82)(S+0.457-j1.82)} \Big|_{S=-0.205-j2.54}$$

or

$$B_i = (0.0220 - j0.0075) (-0.205 - j2.54)^i$$

$$C_i = \frac{S^i}{(S+0.035)(S+0.405+j2.54)(S+0.205-j2.54)(S+0.457-j1.82)} \Big|_{S=-0.457-j1.82}$$

or

$$C_i = (-0.440 + j 0.00226)(-0.457 - j 1.81)^i$$

These numerator coefficient multipliers are presented for $i = 0, 1, 2, 3, 4$ in Figure 3. The partial fractions expansion of

$$\frac{aS^4 + bS^3 + cS^2 + dS + e}{S^5 + 1.36S^4 + 10.47S^3 + 7.88S^2 + 23.3S + 0.81}$$

is then

$$\begin{aligned} & \frac{A_4 a + A_3 b + A_2 c + A_1 d + A_0 e}{S + 0.035} + 2 \operatorname{Re} \left\{ \frac{B_4 a + B_3 b + B_2 c + B_1 d + B_0 e}{S + 0.405 + j 2.54} \right\} \\ & + 2 \operatorname{Re} \left\{ \frac{C_4 a + C_3 b + C_2 c + C_1 d + C_0 e}{S + 0.457 + j 1.81} \right\} \end{aligned}$$

Inversion of the Laplace transforms for any desired vehicle motion is then a matter of finding the transform, breaking it down into partial fractions, and then inverting.

The formulae below shows how the last step is taken.

$$\mathcal{L}^{-1} \left\{ \frac{1}{S+a} \right\} = e^{-at}$$

$$\mathcal{L}^{-1} \left\{ \frac{c+jd}{S+a+jb} \right\} = (c+jd) e^{-(a+jb)t} = (c+jd) e^{-at} (\cos bt - j \sin bt)$$

$$\mathcal{L}^{-1} \left\{ \frac{c-jd}{S+a-jb} \right\} = (c-jd) e^{-(a-jb)t} = (c-jd) e^{-at} (\cos bt + j \sin bt)$$

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{c+jd}{S+a+jb} \right\} + \mathcal{L}^{-1} \left\{ \frac{c-jd}{S+a-jb} \right\} &= \mathcal{L}^{-1} \left\{ \frac{2c(S+a) + 2db}{S^2 + 2aS + a^2 + b^2} \right\} \\ &= e^{-at} (2c \cos bt + 2d \sin bt) \end{aligned}$$

A well-conditioned experiment which could be performed in flight would be one in which only one degree of freedom is disturbed, at least initially. In order to find the stability derivatives of the vehicle influencing its motion in the heave direction, the heave position could be non-aerodynamically maintained at a value different from the equilibrium position. Then the constraint could be removed and the vehicle would return to its equilibrium states in velocity, pitch and heave. Pitch and velocity in the non-equilibrium steady state condition are found by holding heave off equilibrium while eliminating the effects of pitch and velocity feedback to heave, which would bring it to equilibrium. The set of steady state values thus obtained for u , γ , and $\dot{\alpha}$ determine all other derivatives when the heave constraint is removed ($t=0$). Then the well-known initial value theorem and the relations between transforms of a function's derivatives and the transform of the function allow solution for the transform of the function of interest, here $\gamma(t)$. Similar procedures for initial constraint of velocity and pitch produce returns to equilibrium involving mainly the velocity and pitch variables, respectively, and their first and second derivatives. Such experiments are well conditioned for finding the non-coupling derivatives, which are the only derivatives which can be found when only one channel of information is available to the computer.

The problem of making the solved motion of the vehicle available to the computer was solved with the help of a Hewlett-Packard X-Y plotter and line following attachment. This was found to operate well at about one inch per second sweep speed or slower, requiring time scaling on the computer. The analog computer is capable of integrating input voltages well, making the acceleration the natural input variable in each case.

C. The Method of Least Squares

The problem of determining stability derivatives from flight test data is that of finding the set of derivatives which reproduce the data with the least error. The usual error criterion is the square of the difference between the actual and the assumed system output. The problem of minimizing this value is solved by the method of least squares. In case more than one output is involved, a weighted sum of squared errors is minimized, the weights being determined by the relative accuracy of the data.

The data can be either discrete or continuous in nature. As most vehicle attitude sensors, rate meters, and other instrumentation provide continuous data, a certain amount of information is lost when values at selected instants are the only information available. This loss can be made negligible when data intervals are considerably smaller than the periods of vehicle motions, allowing the use of the digital computer in dynamic system analysis.

The computer programs presented in this thesis are derived from the method of steepest descent, a simple yet powerful technique for determining the coefficients of terms in linear equations. The mathematical procedure is shown below, with notation applicable to dynamic systems.

Let

$$\ddot{z} = \sum_{i=1}^n z_{x_i} x_i$$

be the equation to be satisfied by the unknown z_{x_i} . Assume $\ddot{z}(t)$ and all the $x_i(t)$ (state variables responsible for acceleration) are known. The z_{x_i} may be assigned initial values which may or may not be expected to yield small error. In any case, error is defined by

$$\epsilon = \sum \bar{z}_{x_i} x_i - \ddot{z}$$

We seek $\bar{z}_{x_i} \longrightarrow z_{x_i}$, and $\epsilon \rightarrow 0$, or rather error squared (integrated over time) \rightarrow minimum.

Let

$$G = \int_0^t \epsilon^2 dt$$

$$= \int_0^t \epsilon^2 \left(\sum_{i=1}^n \bar{z}_{x_i} x_i - \ddot{z} \right) dt$$

For τ = a computer time variable,

$$\frac{dG}{d\tau} = \sum_{i=1}^n \frac{dG}{d\bar{z}_{x_i}} \frac{d\bar{z}_{x_i}}{d\tau}$$

However

$$\frac{dG}{d\bar{z}_{x_i}} = \int_0^t 2\epsilon \frac{d\epsilon}{d\bar{z}_{x_i}} dt = 2 \int_0^t \epsilon x_i dt$$

Thus G can be forced to a minimum if we choose

$$\frac{d\bar{z}_{x_i}}{d\tau} = - \frac{dG}{d\bar{z}_{x_i}}$$

since then

$$\frac{dG}{d\tau} = - \sum_{i=1}^n \left(\frac{dG}{d\bar{z}_{x_i}} \right)^2 < 0$$

The \bar{z}_{x_i} are found by programming the result

$$\frac{d\bar{z}_{x_i}}{d\tau} = -2 \int_0^t \left(\sum_{i=1}^n \bar{z}_{x_i} x_i - \ddot{z} \right) x_i dt$$

A certain amount of care must be used in programming these equations. The feedback loops required in all but the simplest programs may cause instability due to loop gains higher than unity. The general rule for safe programming is to integrate first and process later, whether the system is simple ($-\epsilon = \ddot{y} - \bar{z}_y y$, Figure 5) or complex

$$(\epsilon = \bar{z}_y y + \bar{z}_{\dot{y}} \dot{y} + \bar{z}_u u + \bar{z}_{\ddot{y}} \ddot{y} + \bar{z}_u u - \ddot{y})$$

Figure 6).

The experimental verification of these formulae requires a certain amount of pre-planning as well as a certain amount of ingenuity. When a computer is capable of handling a program which includes every variable significantly affecting the acceleration under investigation, the feedback operates to bring every coefficient being determined to the value which brings the error very nearly to zero. However, when the computer cannot handle some of the affecting variables, the error in the assumed form of equation cannot be easily reduced. For the second order system

$$\ddot{\gamma} = Z_{\gamma} \gamma + Z_{\dot{\gamma}} \dot{\gamma}$$

the error

$$\epsilon = \bar{Z}_{\gamma} - \ddot{\gamma}$$

(as in the first order program used here) cannot be reduced much below $-Z_{\dot{\gamma}} \dot{\gamma}$. At the same time the computed value of Z_{γ} is affected adversely since any component of $Z_{\dot{\gamma}} \dot{\gamma}$ in phase with γ will be treated as though it were due to an increment in \bar{Z}_{γ} . In Appendix 4 the error in the calculated Z_{γ} (as well as in $Z_{\dot{\gamma}}$, assuming either

$$\epsilon = \bar{Z}_{\dot{\gamma}} \dot{\gamma} - \ddot{Z}, \quad \text{or}$$

$$\epsilon = \bar{Z}_{\dot{\gamma}} \dot{\gamma} - (\ddot{Z} - Z_{\gamma} Z) \quad)$$

is determined and a correction plot is presented.

The problem of determining the stability derivatives of the fifth order system developed in the previous section is even more a problem. First, disturbances were chosen which affected mainly one degree of freedom, to minimize coupling effects. This conditioning of the experiment reduces heave or pitch motion to very nearly second-order-system response (damped sinusoidal oscillation), at least for small time, and it reduces velocity disturbance to nearly a first order system response (decaying exponential).

The effect of coupling is to move the poles of the fifth order system from those of the uncoupled systems, causing here a considerable error in the damping derivatives in heave and pitch. The error in the pole here associated with velocity is less noticeable, and the first order program ($\epsilon = \bar{u}_0 u - \dot{u}$) can be expected to give a very good result.

An innovation promising improved use of available data was attempted but proved to be perhaps more trouble than value, at least for the method of data input used. The unimproved program obtained nearly all the information it could use in the first part of the data run, since the motions were damped and contributed very little to

$$\int_0^t \dot{y}^2 dt \quad (\text{or} \quad \int_0^t \dot{z}^2 dt)$$

and

$$\int_0^t \ddot{y} y dt \quad (\text{or} \quad \int_0^t \ddot{z} \dot{y} dt)$$

after the first period or so. A redefinition of the variables as $z^* = z e^{\alpha' t}$ with α' chosen to approximately cancel the damping rate ($\dot{y}^* = \dot{y} e^{\alpha' t}$, $\ddot{y}^* = \ddot{y} e^{\alpha' t}$), gave the program for $\epsilon = \bar{z}_y z^* - \dot{y}^*$ the form

$$\frac{dZ_y}{d\tau} = -2 \int_0^t \epsilon z^* dt$$

or in steady state
$$Z_y = \frac{\int_0^t \epsilon z^* dt}{\int_0^t z^{*2} dt}$$

where the integrals need not stop increasing at a good rate as time increases. The program based on this technique was very sensitive to input errors as time increased, causing the recorded \dot{y}^* or z^* to depart in the positive or negative sense from sinusoidal curves. Consequently the inte-

grals behaved erratically with time, causing their ratio to depart considerably from the correct value. The results in Figure 7 are for the "unimproved" program (Fig. 5) with "undamped" results for $Z_{\dot{\gamma}}$ and $Z_{\dot{\theta}}$ as well. The "unimproved" program can be expected to give good results for the coefficients $Z_{\dot{\gamma}}$ and $Z_{\dot{\theta}}$, since the initial γ and θ disturbances are the major causes of acceleration for small time.

III. DISCUSSION

The calculated velocity, heave and pitch accelerations for respective initial disturbances were plotted and put on the curve follower. The computer operated on this input from the curve follower to generate the results shown in Figure 7. Numerical results are tabulated in Figure 14.

U_u , the effect of velocity on axial acceleration, is computed by the steps shown in Figure 7a. This first order correlation is very accurate here, where the oscillations associated with the complex poles have been ignored. This corresponds to axial acceleration including these oscillations after they have damped out (see top of Figure 15) and therefore is not an oversimplified demonstration. The result is $\bar{U}_u = -0.035$, corresponding to the coupled system's pole on the real axis. This is nearly $U_u = -0.033$, the value used in the equations of motion. Coupling causes the error of about 6%, which cannot be reduced until coupling derivatives are found. Z_η and $Z_{\dot{\eta}}$, the effects of heave position and its rate of change upon heave acceleration, are computed as shown in Figures 7b and 7c. The graphed outputs are $\bar{Z}_\eta = -7.0$ and $\bar{Z}_{\dot{\eta}} = -0.43$, but voltage meter readings of the outputs varied from -6.4 to -6.8 for \bar{Z}_η and from -0.4 to -0.5 for $\bar{Z}_{\dot{\eta}}$, depending on the instant the computing was stopped and on the accuracy of curve follower calibration. These values lie within 3% and 20% of those indicated for the coupled system (-6.56 and -0.411). The damping derivative $Z_{\dot{\eta}}$ is -0.667 in the equations of motion. The nearly 40% error is due to coupling. The derivative $Z_{\dot{\eta}}$ is -6.27, the error of 10% again due in part to coupling. To reduce error, Z_η was assumed known before $\bar{Z}_{\dot{\eta}}$ was determined.

An attempt to determine the same derivatives by augmenting each variable with time so as to approximately cancel the damping rate is shown in Figures 7d and 7e. As amplitude and phase relations are unaffected, the resulting correlations should give good values for \bar{Z}_y and $\bar{Z}_\dot{y}$, with the possible advantage that motion after the first few seconds is not virtually ignored as in the original program. This "advantage" became a source of error as small discrepancies in curve follower calibration caused large errors in derived variables as time increased. The results $\bar{Z}_y \cong 5$ and $\bar{Z}_\dot{y} = 0.6$ are not as near the expected output as are the results above for the original program. Again, and below as well, the "spring" coefficient was assumed known in order to find the damping derivative.

Finally, $\bar{\Theta}_\mu$ and $\bar{\Theta}_\dot{\mu}$ computer output appears in Figures 7f and 7g. Results seem to be $\bar{\Theta}_\mu = -2.65$ and $\bar{\Theta}_\dot{\mu} = -0.88$, but voltmeter readings indicated $\bar{\Theta}_\mu = -3.38$ and $\bar{\Theta}_\dot{\mu} = -0.94$. The coupled system poles associated with pitch correspond to $\Theta'_\mu = -3.57$ and $\Theta'_\dot{\mu} = -0.914$, for errors of 6% and 3% respectively, but the values of these terms in the equations of motion are $\Theta_\mu = -3.67$ and $\Theta_\dot{\mu} = -0.66$ respectively. The errors are 8% and nearly 40%, where the damping derivative error is nearly entirely due to coupling effects.

An attempt to effectively eliminate damping from the variables involved was rendered hopeless by the high damping rate of the pitch oscillation and the dominance of the heave-associated oscillation, less highly damped, as time increased beyond 12 sec.

The coupling effects on the poles of the fifth order system could have been nearly any effects. Here the major effect was to change the damping ratios of the heave-and-pitch-associated oscillations.

A more complex program such as that shown in Figure 6 would give better results by accounting for all the variables that significantly affect heave acceleration. The input motion should include disturbances in every variable of interest, in order that the effect of each be well defined.

The equations of motion were programmed for computer solution as shown in Figure 12. The same initial conditions used in deriving the Laplace transforms were applied to the appropriate integrators, and computation gave the results in Figure 13. Comparison with the inputs from the curve follower (Figure 7) show that no major errors were committed in deriving the accelerations plotted for curve follower use. The practical difficulties of appropriating two analog computers and guaranteeing that they begin computing simultaneously necessitated the Laplace transform-curve follower approach.

Because the curve follower could not follow the curves more accurately than about 0.03 inch (out of approximately 10 inches maximum sweep, top to bottom), a noise input is present in Figure 7. This has the form of comparatively high frequency, low amplitude oscillations about the correct input, as the line sensor "hunted" for the line. This error is not noticeable after integration, which has a smoothing effect. A more troublesome source of error for this method of data input is bias, or difficulty in calibrating the curve follower output. If the zero voltage position on the slide wire does not coincide with that on the curve plot, there is an error that grows with time when integrated. The resulting drift of derived variables is apparent in several places in Figure 7. This type of error can cause large deviations in the computed stability derivatives. Fortunately,

the same correlation technique which determines the derivatives can be applied to correlate any signal against a constant, the correlation factor representing the bias in the signal. It is then a simple matter to subtract the bias from the signal and base further correlation on the corrected signal. Any bias due to sensor calibration error may even be known from previous testing of the sensor. Then the bias can be corrected for without the need for the above computation.

Such bias terms are accounted for in the work on stability derivative determination referenced in this thesis (Refs. 2 and 5). No bias error was expected in this work. Its form was simple, but its correction would require another multiplier and a few more analog amplifiers which were not available on the TR-20.

IV. CONCLUSIONS

The simple computer program tested was found to give results near to those expected from the coupled system, but the program's neglect of coupling effects results in outputs which have rather large errors. A more complex program for which the minimum of the fit error can be reduced considerably would give outputs much closer to their appropriate values in the equations of motion.

Taylor, Iliff, and Powers (Ref. 8) compared the least squares technique to several others which produce estimates of stability derivatives. In complexity and in accuracy it ranked between simple formulas or analog matching and digital computer methods. Their analysis included noise added to each signal, where each noise affected only its own signal, and not the outputs from sensors measuring integrals or rates of change of that signal. The vehicle tested was represented by a set of equations of motion with specified coefficients, much as was done in this report. The least squares technique gave results ranging from 1% error or less for "well defined" derivatives (those which affected the computer time histories most strongly) to 50% error or more for poorly defined derivatives. The best technique presented, the modified Newton-Raphson method, gave results within 10% for the poorly defined derivatives, and results for all derivatives closer to the actual values in the equations of motion than those of the least squares method.

The choice of a data reduction technique this depends on the required accuracy of results and on the availability of computing aids. Barring digital computer methods, there is no technique presently available which gives stability derivatives nearer to the correct values than the analog regression technique of least squares.

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APPENDICES

and

FIGURES

APPENDIX A

A METHOD OF HANDLING NONLINEARITIES

A non-linear function $\eta(\theta)$ may be approximated by an amplitude dependent transfer function, known as the describing function. It is generally a complex number,

$$N(|\theta|, \omega) = (N_r + jN_i)(|\theta|, \omega)$$

where

$$\theta = |\theta| \cos \omega t$$

produces an input $|\theta| (N_r \cos \omega t - N_i \sin \omega t)$, with amplitude and phase dependent on $|\theta|$ as well as ω . N_r and N_i are averages over one period of the motion:

$$N_r = \frac{1}{\pi} \int_0^{2\pi} \eta(\theta(t)) \cos \omega t d(\omega t)$$

$$N_i = -\frac{1}{\pi} \int_0^{2\pi} \eta(\theta(t)) \sin \omega t d(\omega t)$$

It should be possible to invert the process and find $\eta(\theta)$ from data yielding $N(|\theta|, \omega)$. Just such data is obtained for vehicle stability derivatives such as \bar{Z}_2 , which depends on the amplitude of the heave disturbance. Thus amplitude response data could be analyzed to yield the approximate nonlinear dependence of lift on pad clearance.

APPENDIX B

GRAPHICAL FACTOR OF A FOURTH ORDER POLYNOMIAL, AND IMPROVEMENT

The equations presented in the text can be plotted as follows:

$$a_1 = \frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 \left(-c - b - \frac{E}{b_1}\right)}$$

or, $a_1 = 0.662 \pm \sqrt{0.44 - (10.424 - b_1 - \frac{23.04}{b_1})}$ (B-1)

$$a_1 = \frac{D - B b_1}{\frac{E}{b_1} - b_1}$$

or, $a_1 = \frac{7.415 - 1.325 b_1}{\frac{23.04}{b_1} - b_1}$ (B-2)

b_1	a_1 (B-1)	a_1 (B-2)
6.30	(not real)	0.354
6.35	~ 0.662	0.360
6.40	0.49	--
6.45	---	0.395
6.50	0.366	
6.75	0.191	

The plot in Figure 8 shows that the intersection of these two curves lies approximately at $(a_1, b_1) = (0.40, 6.47)$. These roots were improved by the division which follows

$$DEN. = S^2 + (0.40 + \delta)S + (6.47 + \epsilon)$$

$$\begin{array}{r}
 S^2 + (0.925 - \delta)S + (3.584 - \epsilon - 0.525\delta) \\
 \hline
 N) \begin{array}{r}
 S^4 + 1.325S^3 + 10.424S^2 + 7.415S + 23.04 \\
 S^4 + (0.40 + \delta)S^3 + (6.47 + \epsilon)S^2 \\
 \hline
 (0.925 - \delta)S^3 + (3.954 - \epsilon)S^2 + 7.415S \\
 (0.925 - \delta)S^3 + (0.37 + 0.525\delta)S^2 + (5.99 - 6.47\delta + 0.925\epsilon)S \\
 \hline
 (3.584 - \epsilon - 0.525\delta)S^2 + (1.425 + 6.47\delta - 0.925\epsilon)S + 23.04 \\
 \hline
 (3.584 - \epsilon - 0.525\delta)S^2 + (1.435 + 3.37\delta - 0.4\epsilon)S + 23.2 - 2.89\epsilon - 3.4\delta
 \end{array}
 \end{array}$$

$$R = (0)S + (0) = (-0.10 + 3.1\delta - 0.525\epsilon)S + (0.16 + 2.89\epsilon + 3.4\delta)$$

$$3.1\delta - 0.525\epsilon = 0.10$$

$$2.89\epsilon + 3.4\delta = 0.16$$

which when solved gives $\delta = 0.011$, $\epsilon = 0.04$

(this is not quite exact, but the results are acceptable). Check:

$$\begin{array}{r}
 S^2 + 0.411S + 6.51 \\
 S^2 + 0.914S + 3.54 \\
 \hline
 S^4 + 0.411S^3 + 6.51S^2 \\
 0.914S^3 + 0.38S^2 + 5.95S \\
 \hline
 3.54S^2 + 1.46S + 23.04 \\
 \hline
 S^4 + 1.325S^3 + 10.43S^2 + 7.41S + 23.04
 \end{array}$$

APPENDIX C

SOLUTION FOR VEHICLE MOTION

Steady state conditions reduce the system block diagram to the state shown in Figure 9. Assume $z(0)=1.00$, $\dot{z}(0)=0$. This heave initial disturbance also generates disturbances in pitch and velocity

$$\ell(0) = 0.0403, \quad \dot{\ell}(0) = 0$$

$$\psi(0) = -3.77, \quad \dot{\psi}(0) = 0$$

This collection of initial values and the equations of motion determine all other derivatives at $t = 0$:

$$\ddot{z}(0) = -6.69, \quad \ddot{\psi}(0) = 4.46, \quad \ddot{\ell}(0) = 39.29$$

$$\ddot{\psi}(0) = 0.485, \quad \ddot{\ell}(0) = 0.504$$

$$\ddot{\ell}(0) = 0, \quad \ddot{\psi}(0) = 1.32$$

The Laplace transform for the ensuing heave motion $z(t)$ is then found by applying

$$\mathcal{L}[\dot{a}(t)] = s\mathcal{L}[a(t)] - a(0)$$

$$a(0) = \lim_{s \rightarrow \infty} [s a(s)]$$

to the successive derivatives. The result is

$$z(s) = \frac{s^4 + 1.36s^3 + 3.78s^2 + 3.24s - 1.35}{s^5 + 1.36s^4 + 10.47s^3 + 7.88s^2 + 23.3s + 0.81}$$

or, expanded,

$$z(s) = \frac{-0.064}{s+0.035} + \frac{0.908s - 0.083}{(s+0.205)^2 + (2.54)^2} + \frac{0.160s + 0.625}{(s+0.457)^2 + (1.82)^2}$$

Then by usual inverse transform methods

$$\begin{aligned} z(t) = & -0.064 e^{-0.035t} \\ & + e^{-0.205t} (0.908 \cos 2.54t - 0.106 \sin 2.54t) \\ & + e^{-0.457t} (0.160 \cos 1.82t + 0.302 \sin 1.82t) \end{aligned}$$

which twice differentiated becomes

$$\begin{aligned} \ddot{z}(t) = & -0.00008 e^{-0.035t} \\ & + e^{-0.205t} (-5.70 \cos 2.54t + 1.633 \sin 2.54t) \\ & + e^{-0.457t} (-1.00 \cos 1.82t - 0.674 \sin 1.82t) \end{aligned}$$

As a check on the computations,

$$z(0) = 1.004 \quad , \quad \ddot{z}(0) = -6.70$$

which shows good accuracy. $\ddot{z}(t)$ was tabulated and plotted for curve follower application, and appears in the computer output (Figure 7), as do $\dot{z}(t)$ and $z(t)$, which were similarly obtained. When flight test data is available, the inputs to the computer will come in by separate channels, either directly from sensors on the vehicle, from some multivariable equivalent to the curve follower, or from digital-analog conversion devices attached to a digital computer programmed

to solve for the variables of motion from flight test data.

For $\psi(0) = 100$, $\dot{\psi}(0) = 0$

$$\begin{aligned} \ddot{\psi}(t) = & 0.00003 e^{-0.035t} \\ & + e^{-0.205t} (-0.535 \cos 2.54t - 0.389 \sin 2.54t) \\ & + e^{-0.457t} (-3.01 \cos 1.82t + 0.109 \sin 1.82t) \end{aligned}$$

$$\begin{aligned} \dot{\psi}(t) = & 0.0273 e^{-0.035t} \\ & + e^{-0.205t} (0.072 \cos 2.54t + 0.072 \sin 2.54t) \\ & + e^{-0.457t} (+0.906 \cos 1.82t + 0.134 \sin 1.82t) \end{aligned}$$

and for $\psi(0) = 1$, after transients associated with pitch and heave have died out,

$$\psi(t) \cong e^{-0.035t}$$

$$\dot{\psi}(t) \cong -0.035 e^{-0.035t}$$

APPENDIX D

ALTERNATE REGRESSION TECHNIQUES

When no computing aids are available, a rapid estimate of certain stability and control derivatives can be obtained by taking ratios of measured in-flight state variables, if all contributions to the measured output due to causes other than the measured input are negligible. For control response, this could involve measuring the rate of departure of a variable from its steady-state motion after a control pulse or step. For free heave oscillation data this might be $\frac{\ddot{y}(0)}{\dot{y}(0)} \cong Z_y$.

The values obtained for $\ddot{y}(0)$ and $\dot{y}(0)$ were -6.70 and 1.00 with a ratio of -6.70, while $Z_y = -6.27$, for only 7% error. This technique is very simple but requires highly conditioned experiments, which may be difficult to set up. Wind tunnel results may give values for derivatives difficult to obtain in flight, but their accuracy is questionable.

When an analog computer is available, the equations of motion can be programmed and solved for as many variables as may be desirable, using best estimates for the stability derivatives to determine potentiometer settings. Manual adjustment of these settings and visual comparison of the computer output (which may be time scaled to leave a continuous trace on an oscilloscope) to flight test data allows an improved fit to be made. Derivatives which have little effect on the motions of interest can be left at their previously estimated values, while those needing adjustment are determined from the potentiometer settings for best fit. The skill of the operator greatly influences the results.

A more automatic method requiring a larger analog computer to handle a similar number of derivatives is the least squares method used in its

simplest form in this report. Its problems include sensitivity to both measurement noise and non-ideally conditioned experiments.

A similar method exists in which a "method function" multiplies each of the measured quantities, giving their values a weight dependent on time. The results of this technique are little different from those obtained by the least squares method (Ref.2).

Methods better suited to the digital computer are the gradient method and the Newton-Raphson method, which involve iteration. The first is a first order method, while Newton-Raphson uses an approximation to the second gradient which greatly improves convergence. A fit is made to any number of state variables, the weight of each determined by its estimated noise level. The results are generally superior to those obtained by any other method (Ref. 2).

APPENDIX E
SYSTEMATIC ERROR CORRECTION

As an example of the errors associated with the determination of stability derivatives by an oversimplified least squares program, the ideal fit to a second order system response (damped sinusoidal oscillation) by a first order computer program is shown and some conclusions relating to higher order systems are reached.

For

$$z = e^{a't} \sin \beta t$$

$$\dot{z} = \beta e^{a't} \cos \beta t + a' e^{a't} \sin \beta t$$

$$\ddot{z} = -\beta^2 e^{a't} \sin \beta t + a' \beta e^{a't} \cos \beta t + a' \beta e^{a't} \cos \beta t + a'^2 e^{a't} \sin \beta t$$

or

$$\ddot{z} = (a'^2 - \beta^2) e^{a't} \sin \beta t + 2 a' \beta e^{a't} \cos \beta t$$

or

$$\ddot{z} = -(a'^2 + \beta^2) z + 2 a' \dot{z}$$

which has the form

$$\ddot{z} = z_2 z + z_1 \dot{z}$$

$$z_2 = -(a'^2 + \beta^2)$$

$$z_1 = 2 a'$$

A first order program will compare \ddot{z} and z only, and determine the component of \ddot{z} which has the same phase as z (in terms of rotating vectors, that component of \ddot{z} which lies in the direction of z , as shown in Figure 10). This component is $(a'^2 - \beta^2) e^{a't} \sin \beta t$

which when compared to $z = e^{\alpha t} \sin \beta t$ results in a ratio of $\alpha'^2 - \beta^2$ rather than $-\alpha'^2 - \beta^2 = Z_z$, i.e., $Z_z \rightarrow (\alpha'^2 - \beta^2) = Z_z + \frac{1}{2} Z_{ij}$ where Z_{ij} is unknown. Clearly, accurate results for Z_z depend on Z_{ij} being small.

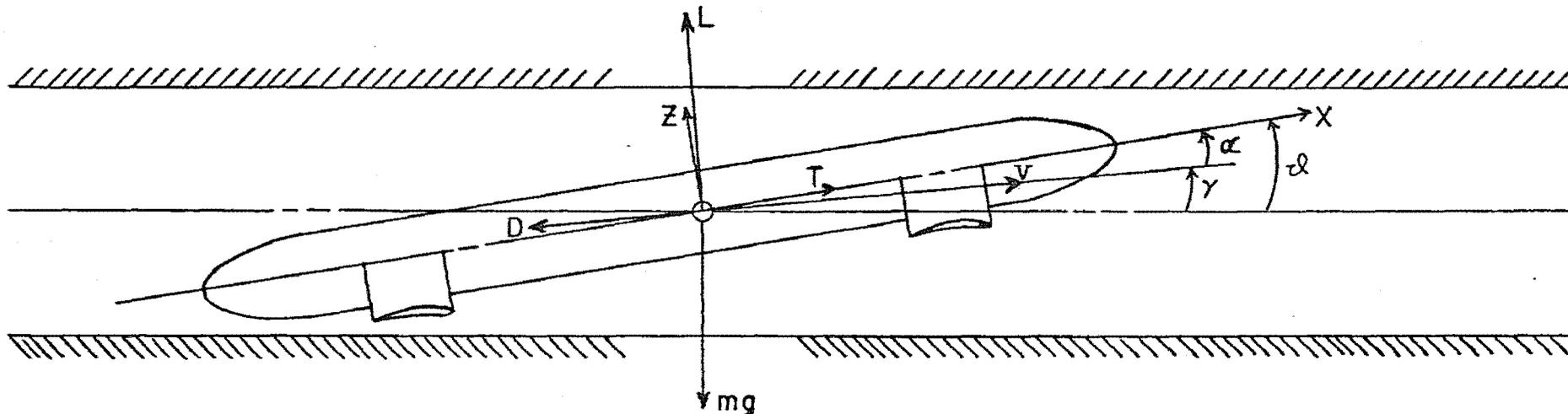
A first order program will find Z_{ij} in a similar manner and the computed result is $\bar{Z}_{ij} \rightarrow \frac{1}{2} Z_{ij}$. This result allows the correct values to be found as

$$Z_{ij} = 2 \bar{Z}_{ij}, \quad Z_z = \bar{Z}_z + 2 \bar{Z}_{ij}^2$$

When Z_{ij} is known exactly the problem is reduced to a first order correlation. In Figure 11, $\bar{\bar{Z}}_{ij} = Z_{ij} + \frac{1}{4} \frac{Z_{ij}}{Z_z}$ is the result of correlating y with $y - \bar{Z}_z z$.

$$\bar{\bar{S}} = \frac{\bar{\bar{Z}}_{ij}}{2\sqrt{Z_z}}, \quad S = \frac{Z_{ij}}{2\sqrt{Z_z}}$$

The corrections mentioned above may have analogs for higher order systems, but the presence of "noise" in the input may result in corrected stability derivatives which are no longer best in the mean square sense. The use of a more complete analog computer program (which eliminates very small derivatives, when the program is limited in size) is preferable.



V ROTATED α CLOCKWISE FROM X,Z RESPECTIVELY.

TUBE AXIS ROTATED ϱ CLOCKWISE FROM Z,X RESPECTIVELY.

Figure 1. Tubeflight Vehicle Axes and Orientation

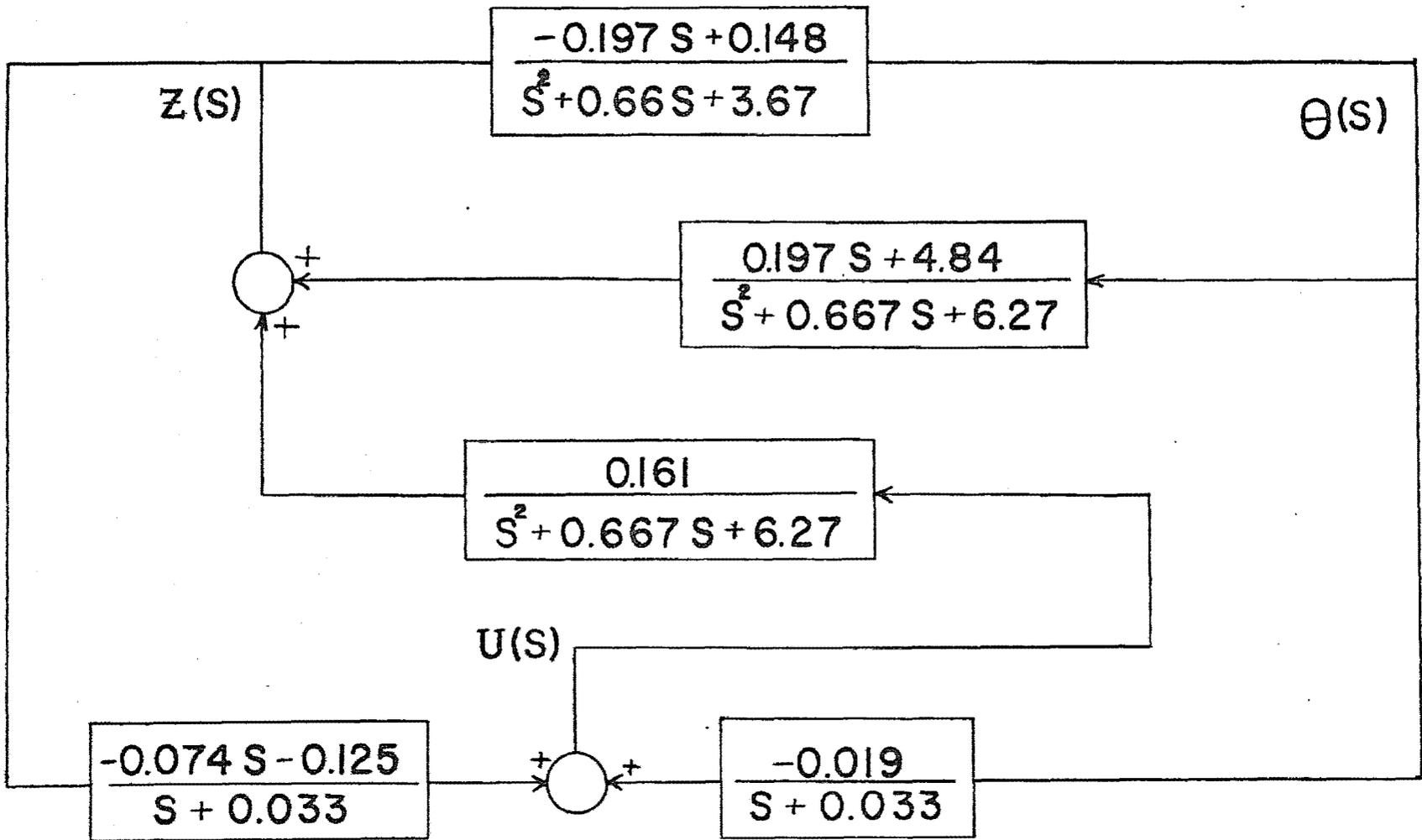


Figure 2. Longitudinal Dynamics Block Diagram

	A_i	B_i	C_i
s^0 $i=0$	+0.044	+0.0220 - 0.0075 j	-0.0440 - 0.00226 j
s^1 $i=1$	-0.00154	-0.0235 - 0.0544 j	+0.0242 + 0.0791 j
s^2 $i=2$	+0.000054	-0.133 + 0.071 j	+0.133 - 0.0805 j
s^3 $i=3$	-0.0000019	+0.207 + 0.324 j	-0.207 - 0.205 j
s^4 $i=4$	+0.000000066	+0.781 - 0.596 j	-0.278 + 0.471 j

Figure 3. Partial Fractions Numerator Coefficient Multipliers

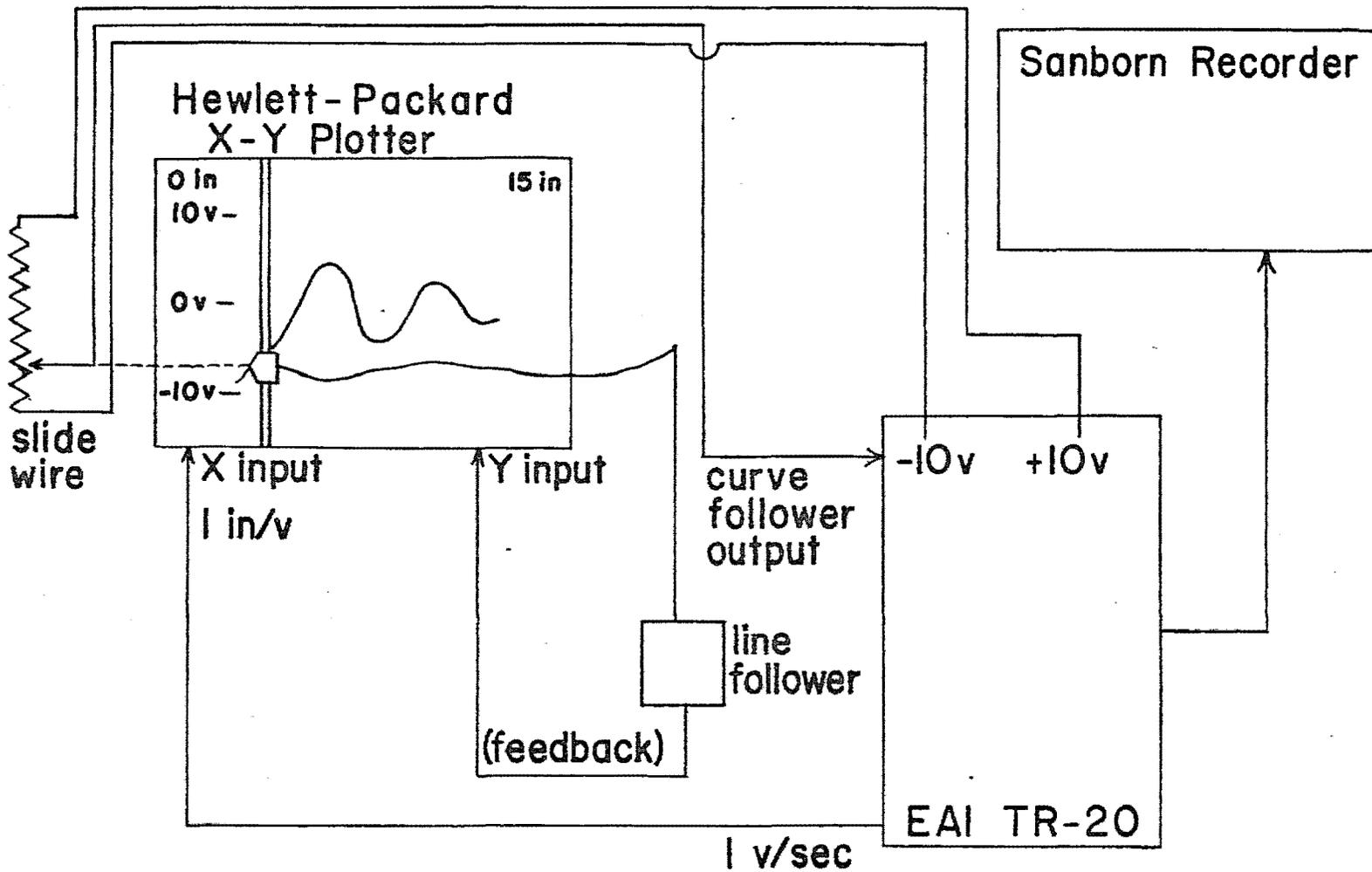


Figure 4. Schematic of Test Set-Up

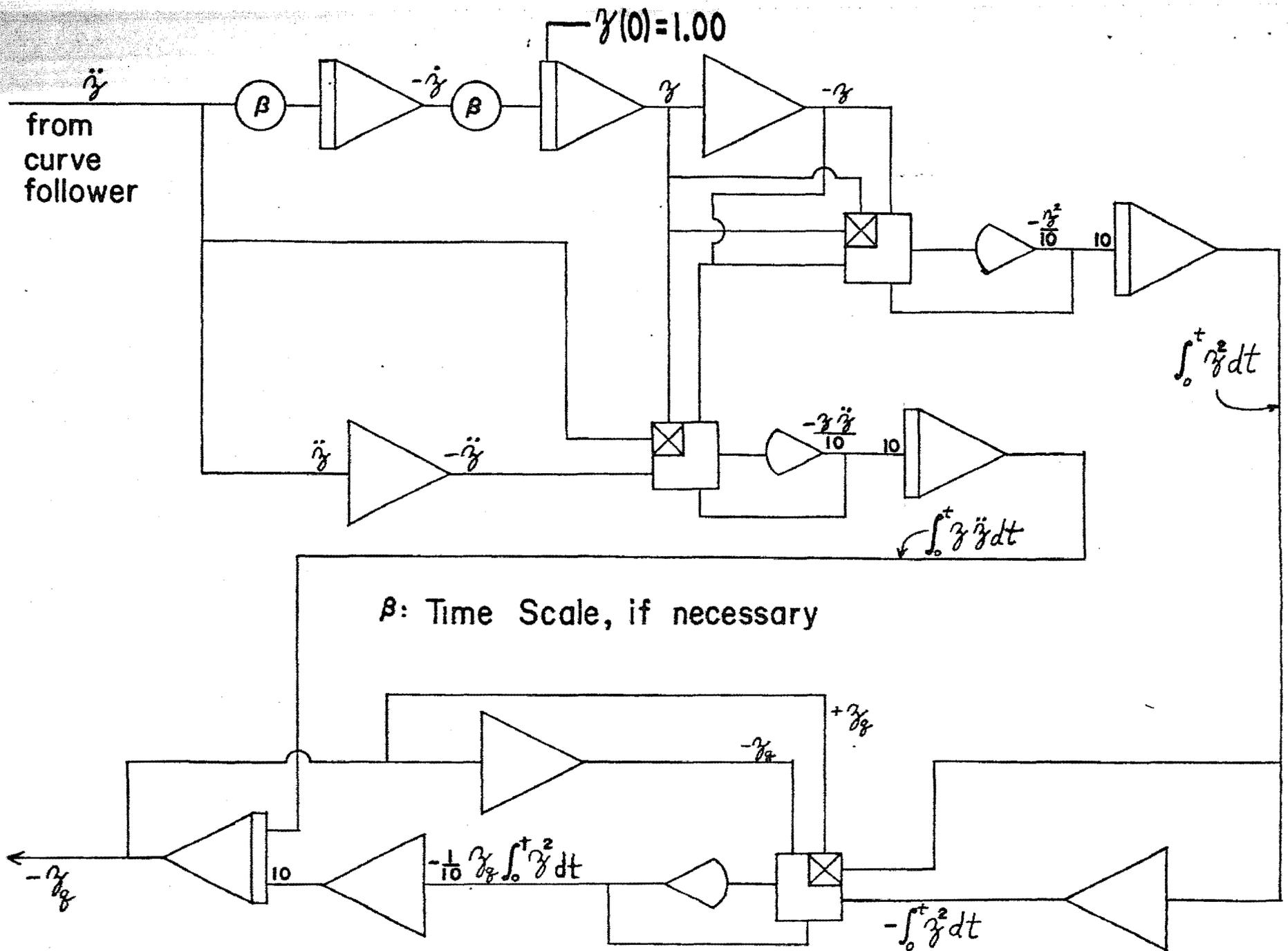


Figure 5. Tested Analog Program

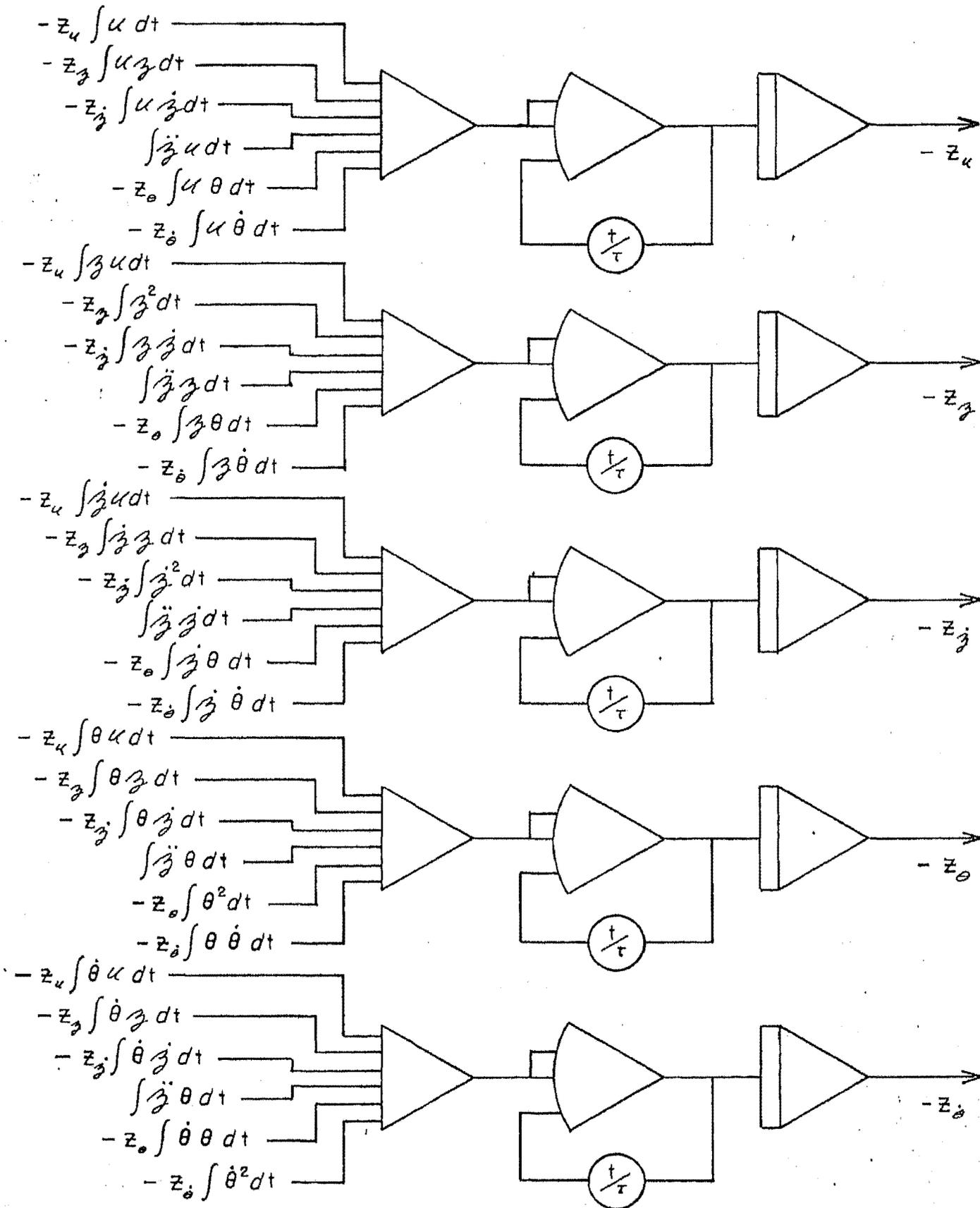


Figure 6. Complete Heave Derivatives Program

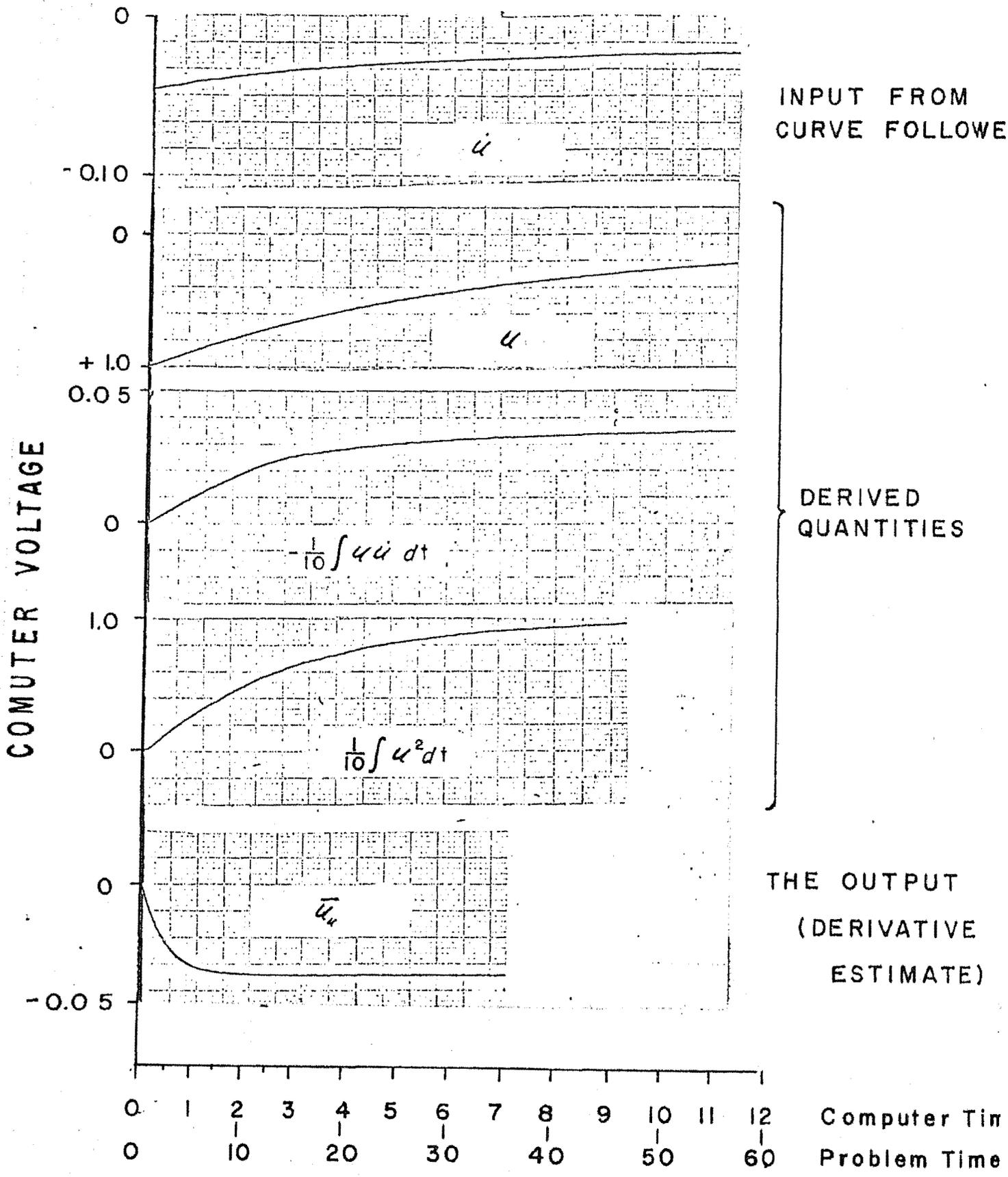


Figure 7a. u_u

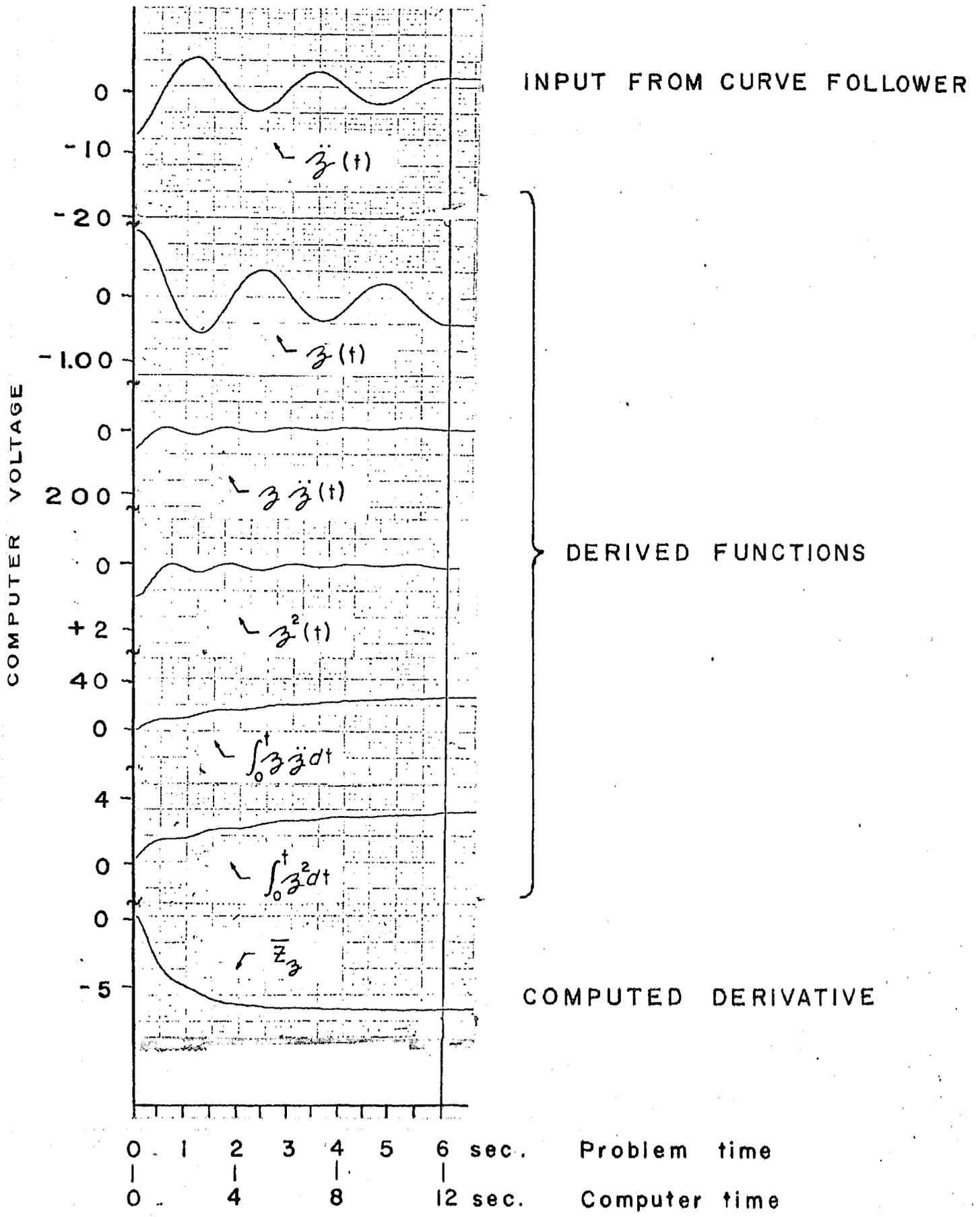


Figure 7b. z_3

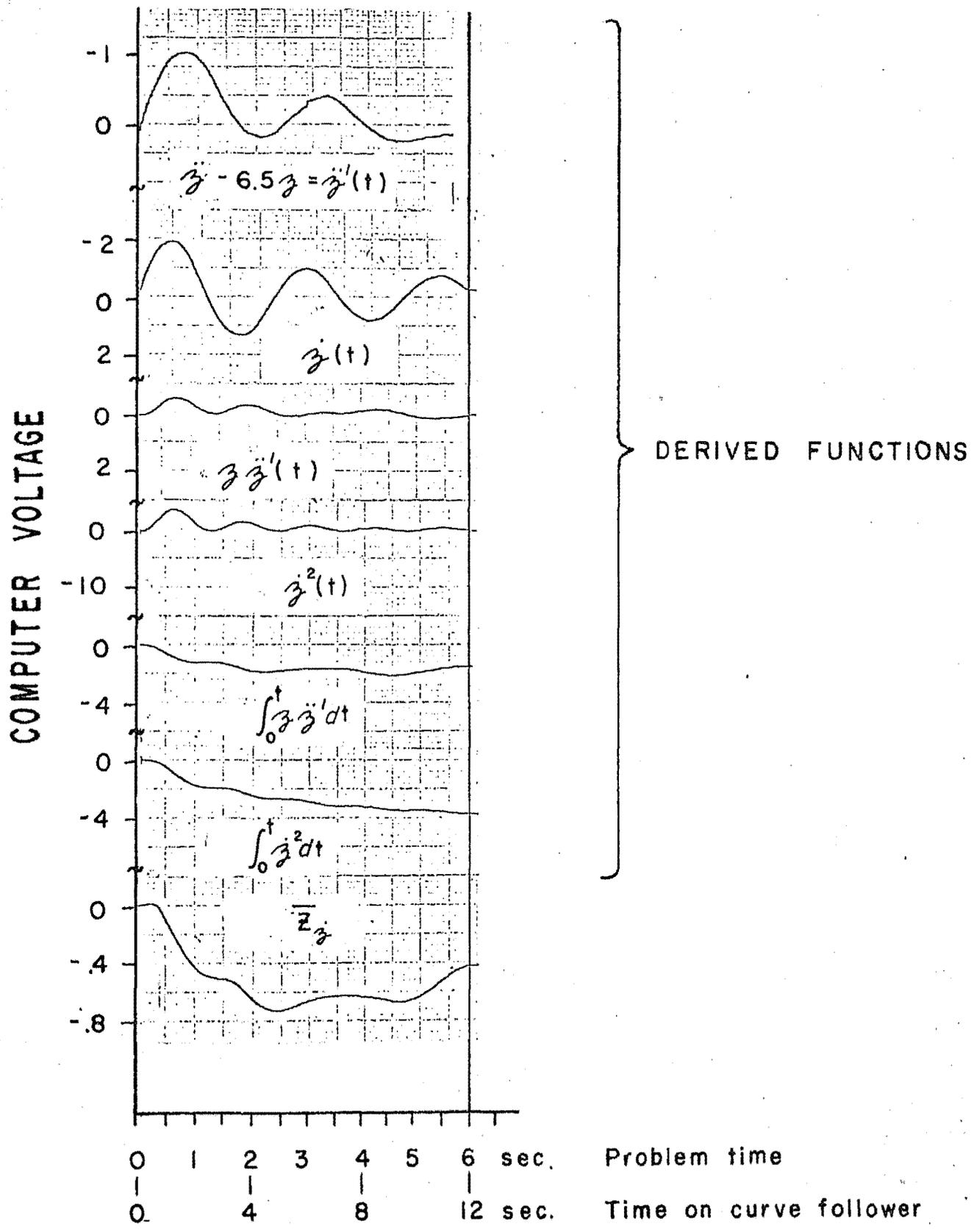
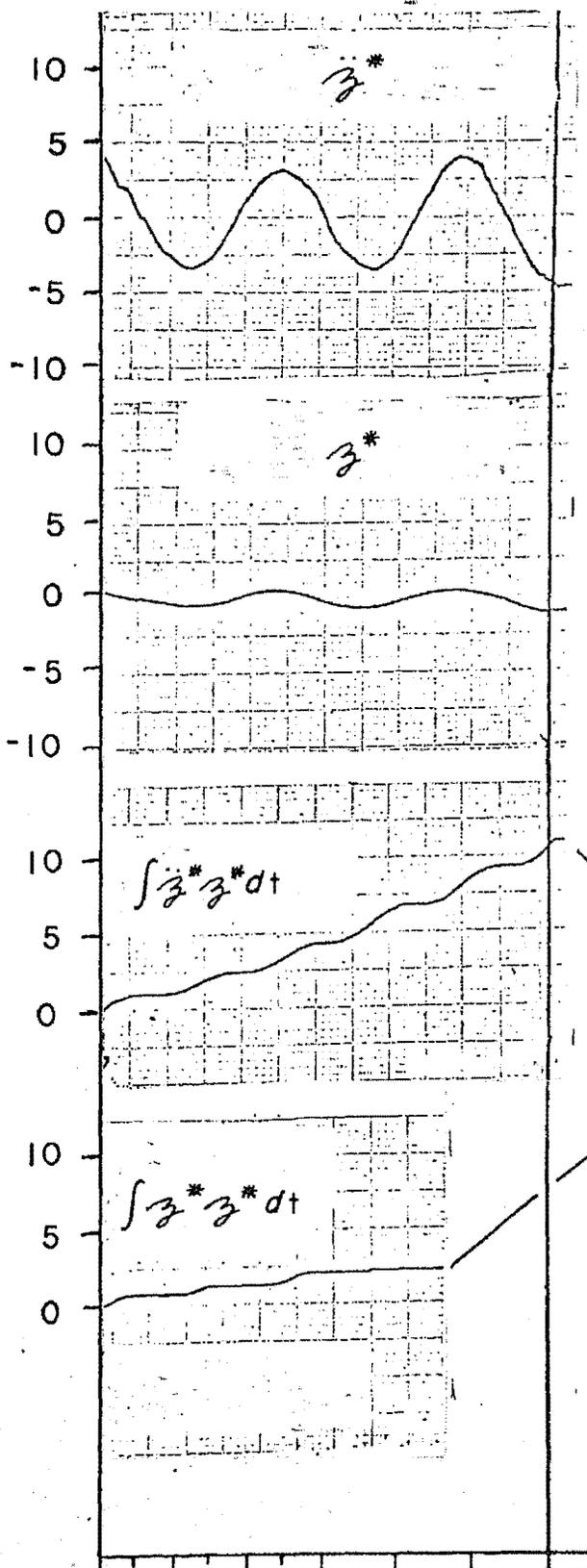


Figure 7c. \bar{z}_z

COMPUTER VOLTAGE



"UNDAMPED" ACCELERATION

"UNDAMPED" POSITION

RATIO OF FINAL VALUES
GIVES ESTIMATE OF
STABILITY DERIVATIVE

$$\text{RATIO} \approx \frac{22}{4.4} \approx 5 \approx \bar{z}_2$$

0 1 2 3 4 5 6 sec. Problem time
0 4 8 12 sec. Computer time

Figure 7d. "Undamped" z_2

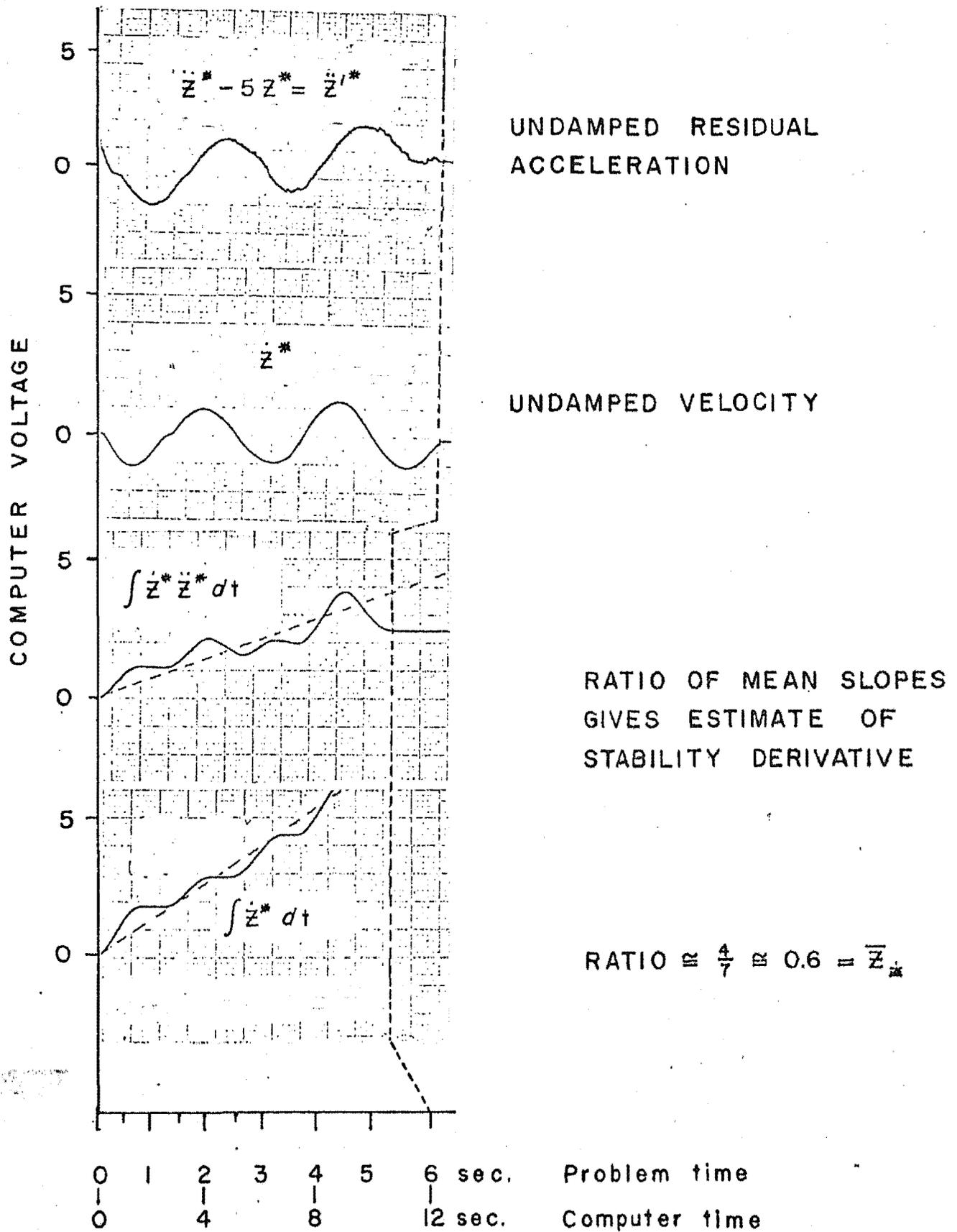


Figure 7e. Undamped \dot{z}_i

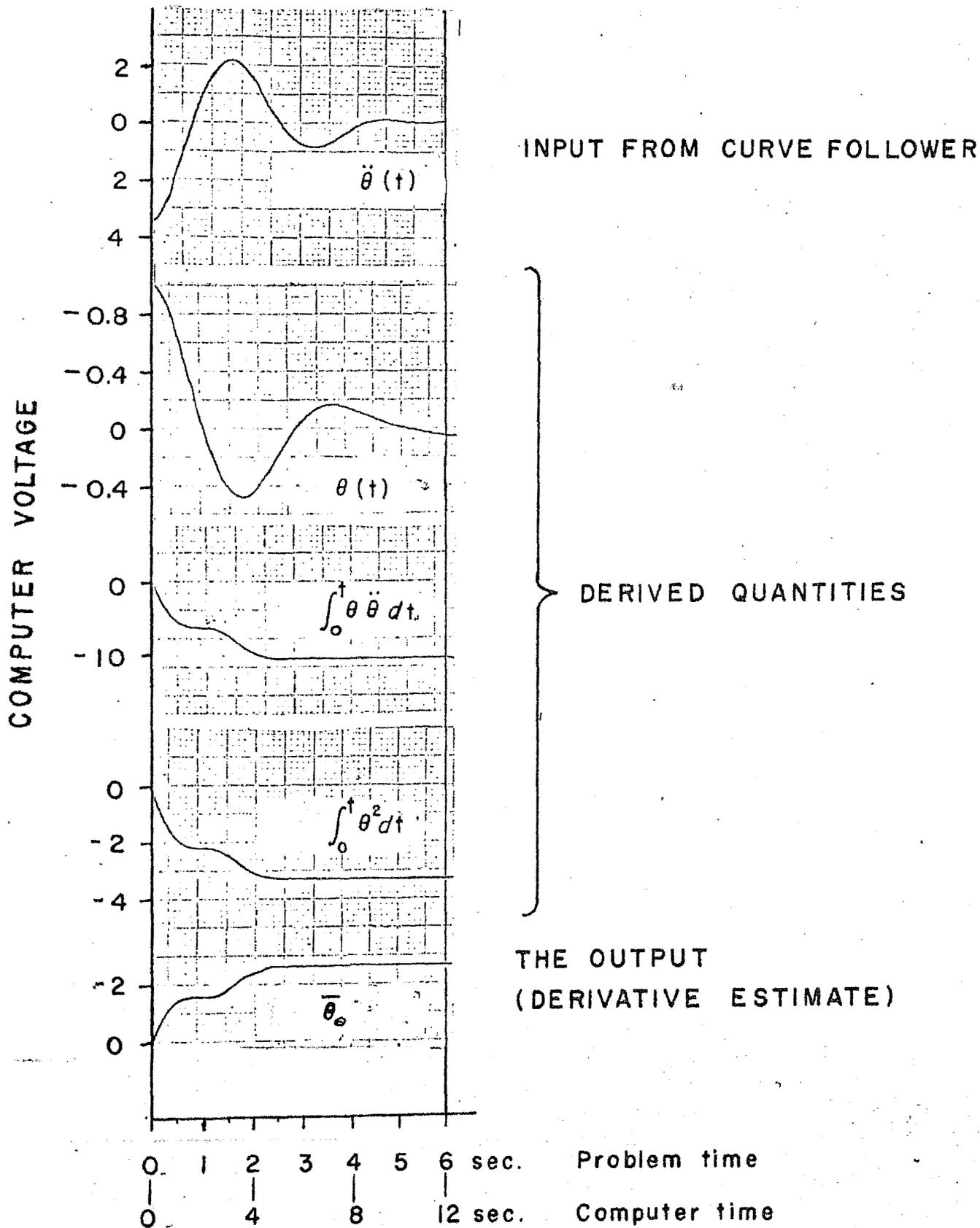
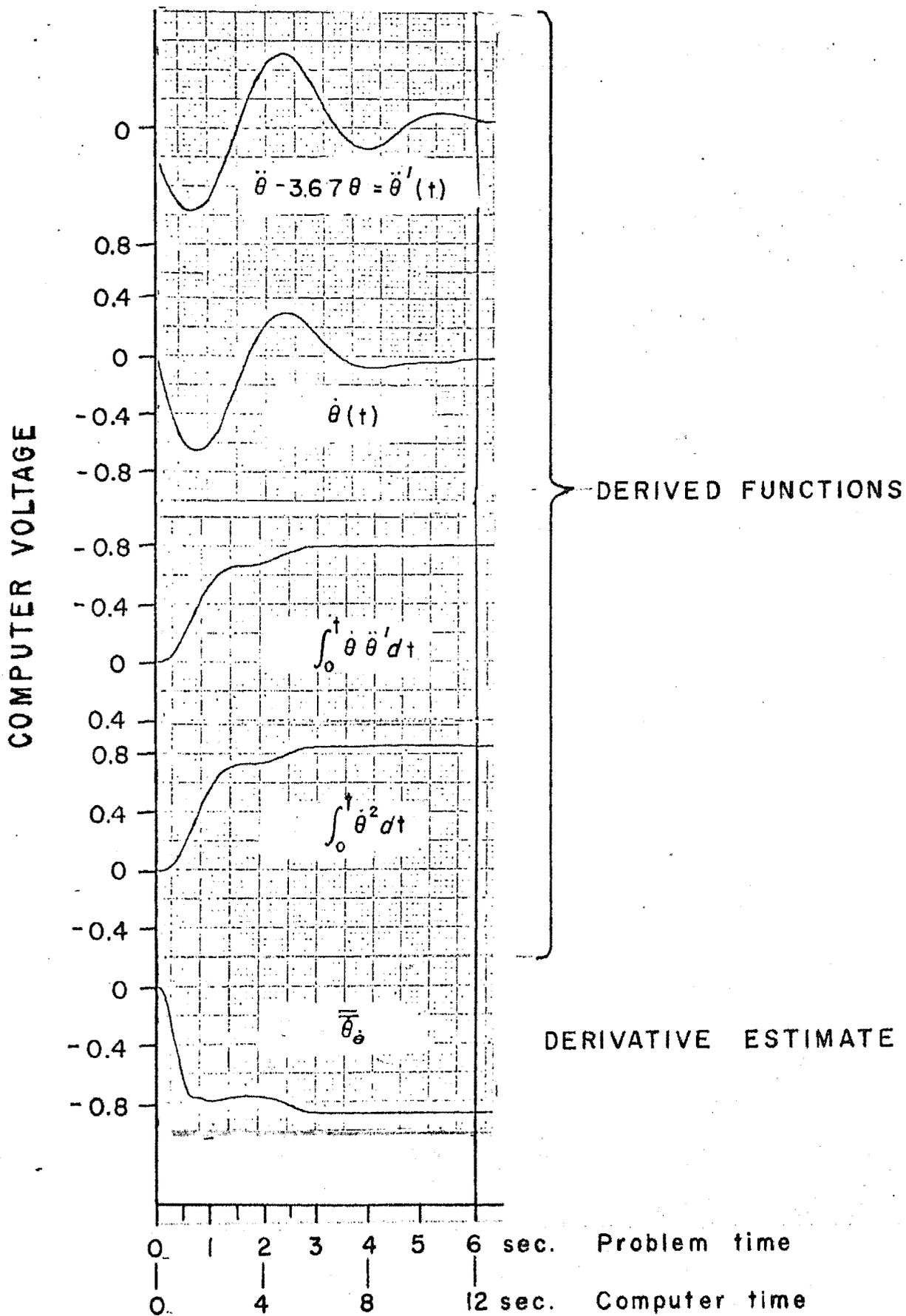


Figure 7f. $\dot{\theta}_e$

Figure 7g. $\overline{\dot{\theta}}$

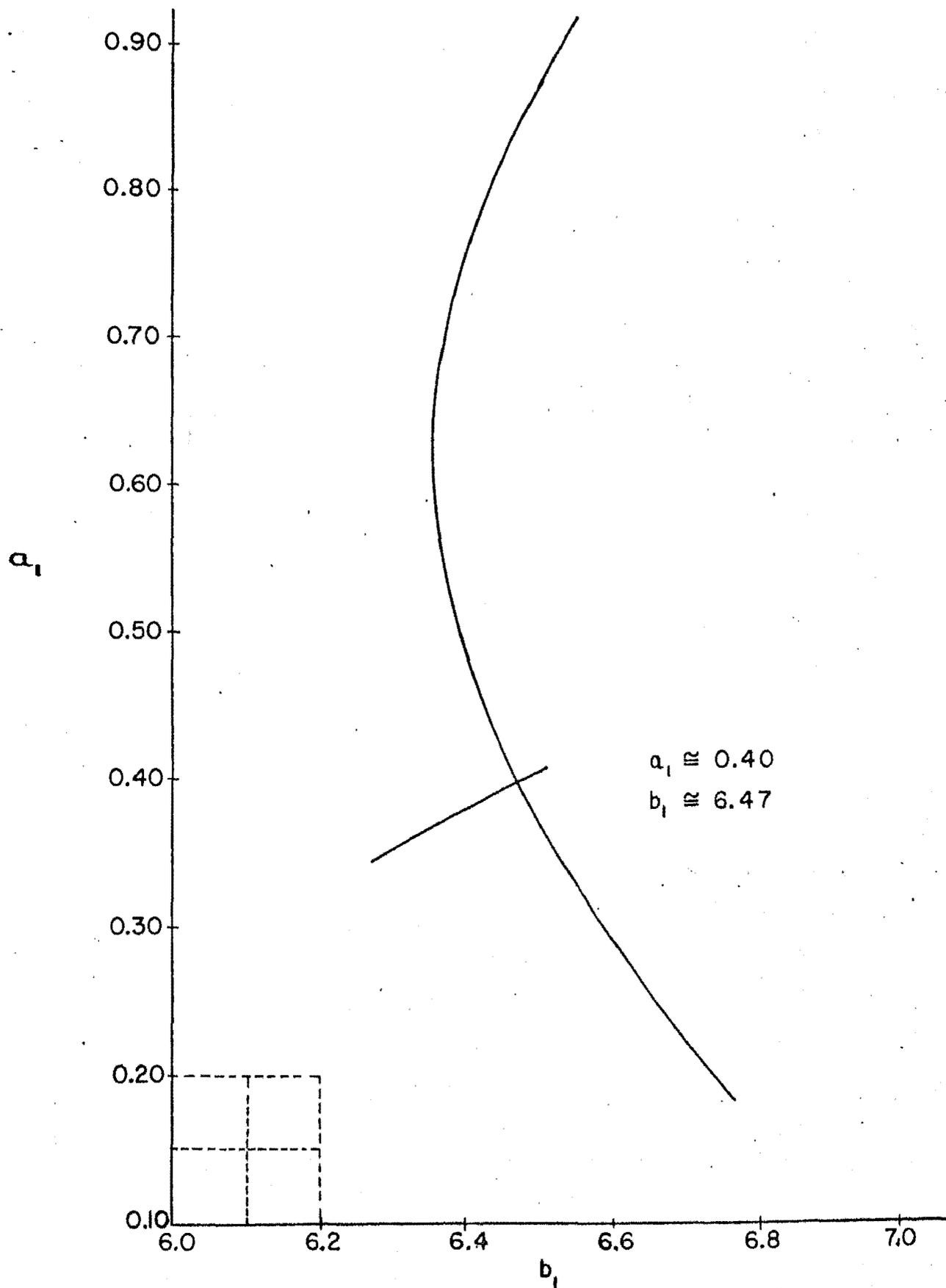


Figure 8. Graphical Factoring of a Fourth Order Polynomial

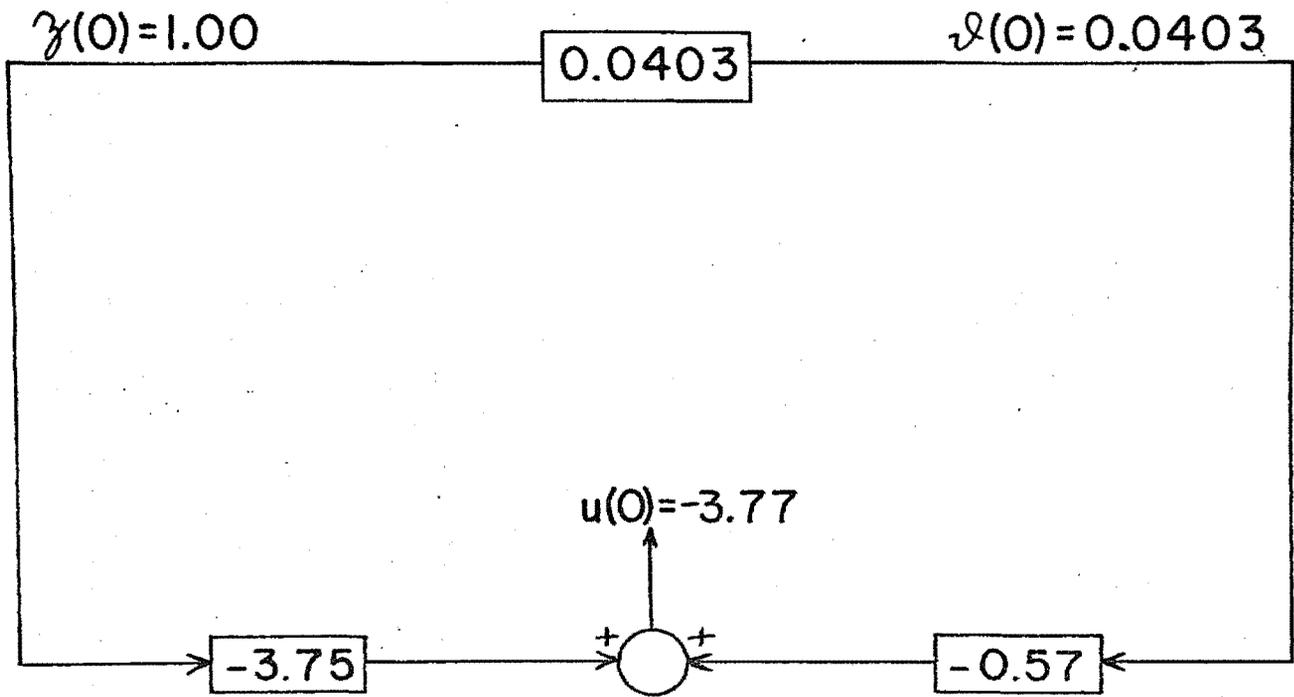


Figure 9. Initial Conditions for Heave Disturbance

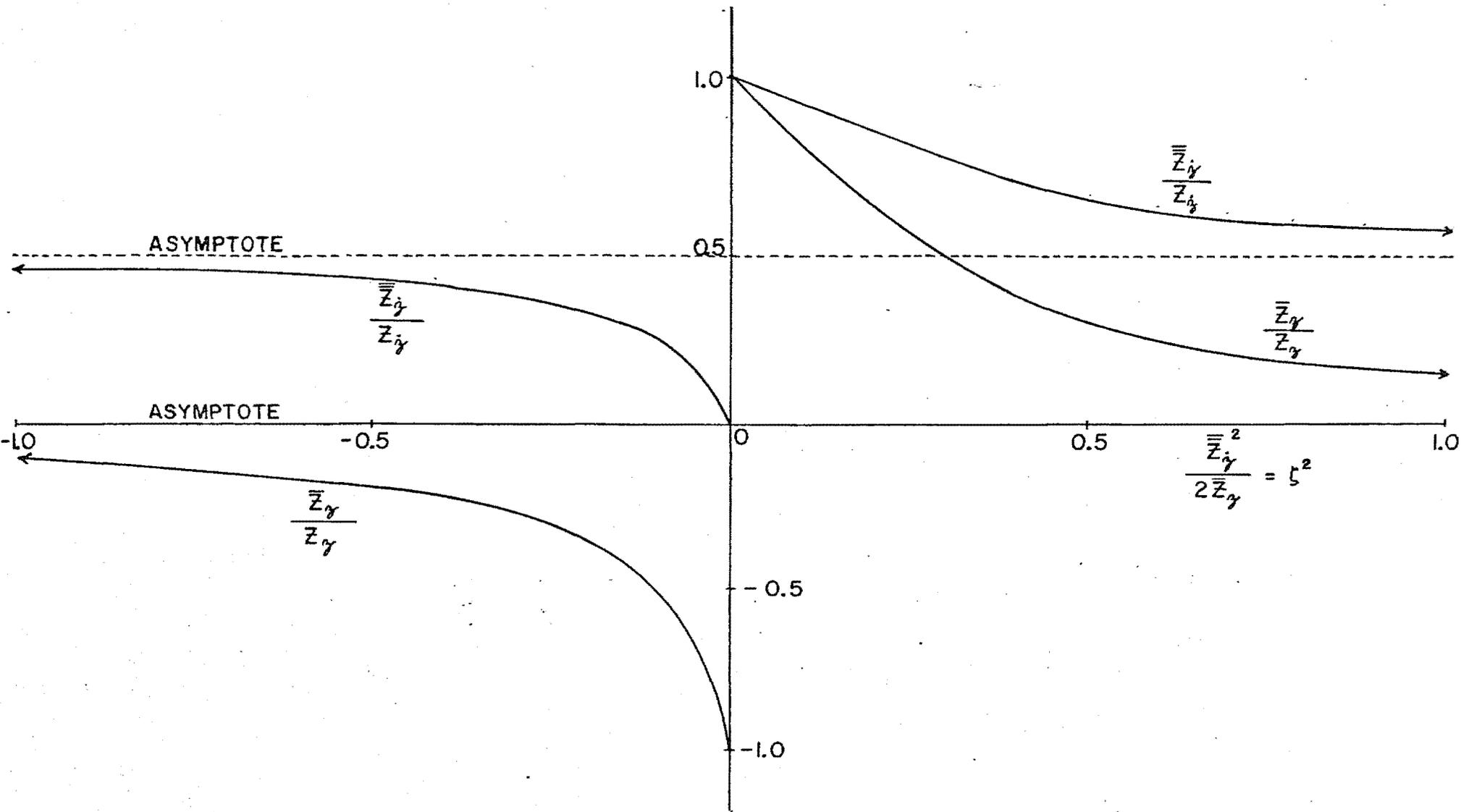


Figure 11. Correction Curves for Coefficients of Second Order System

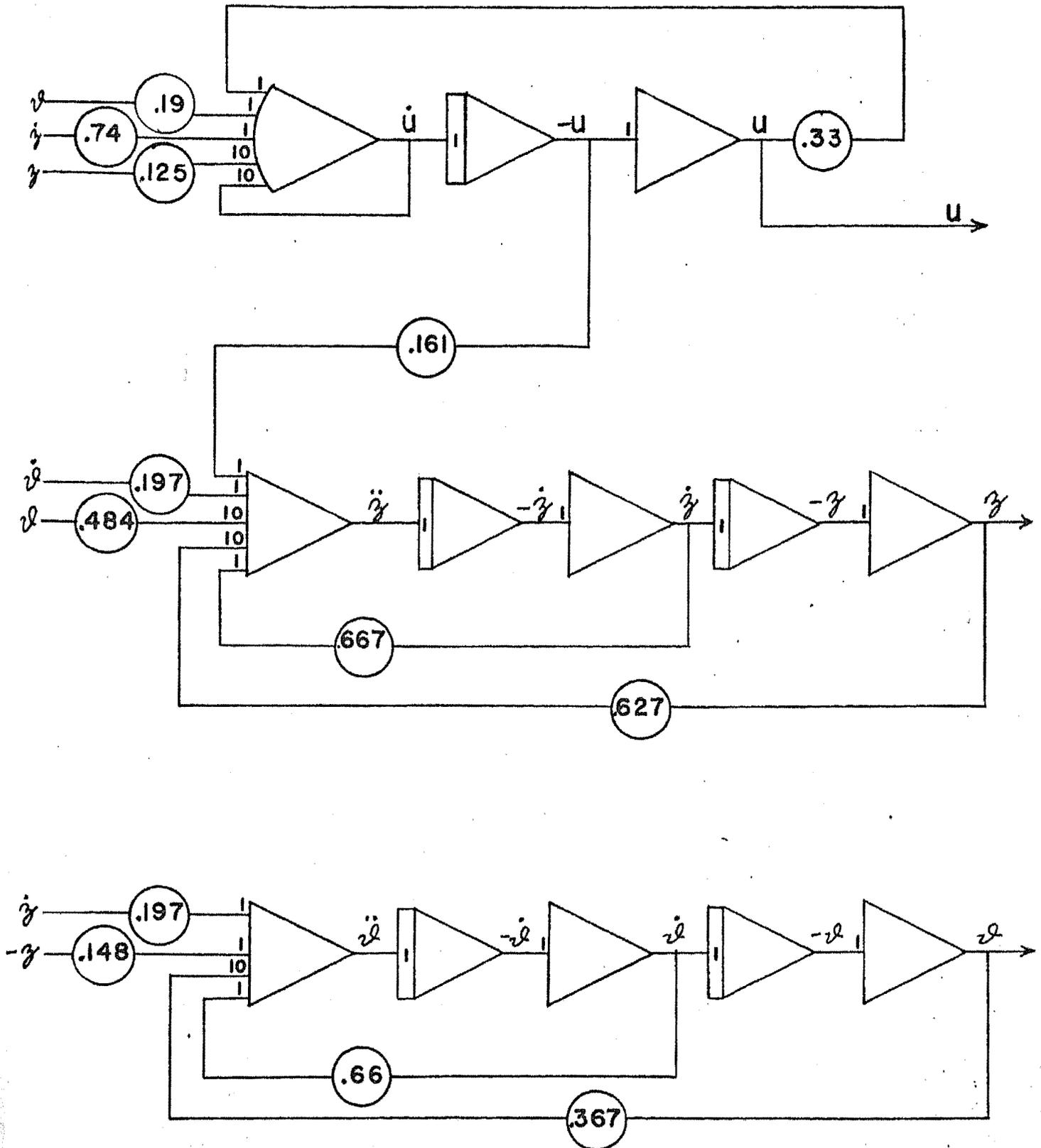


Figure 12. Simulator Program

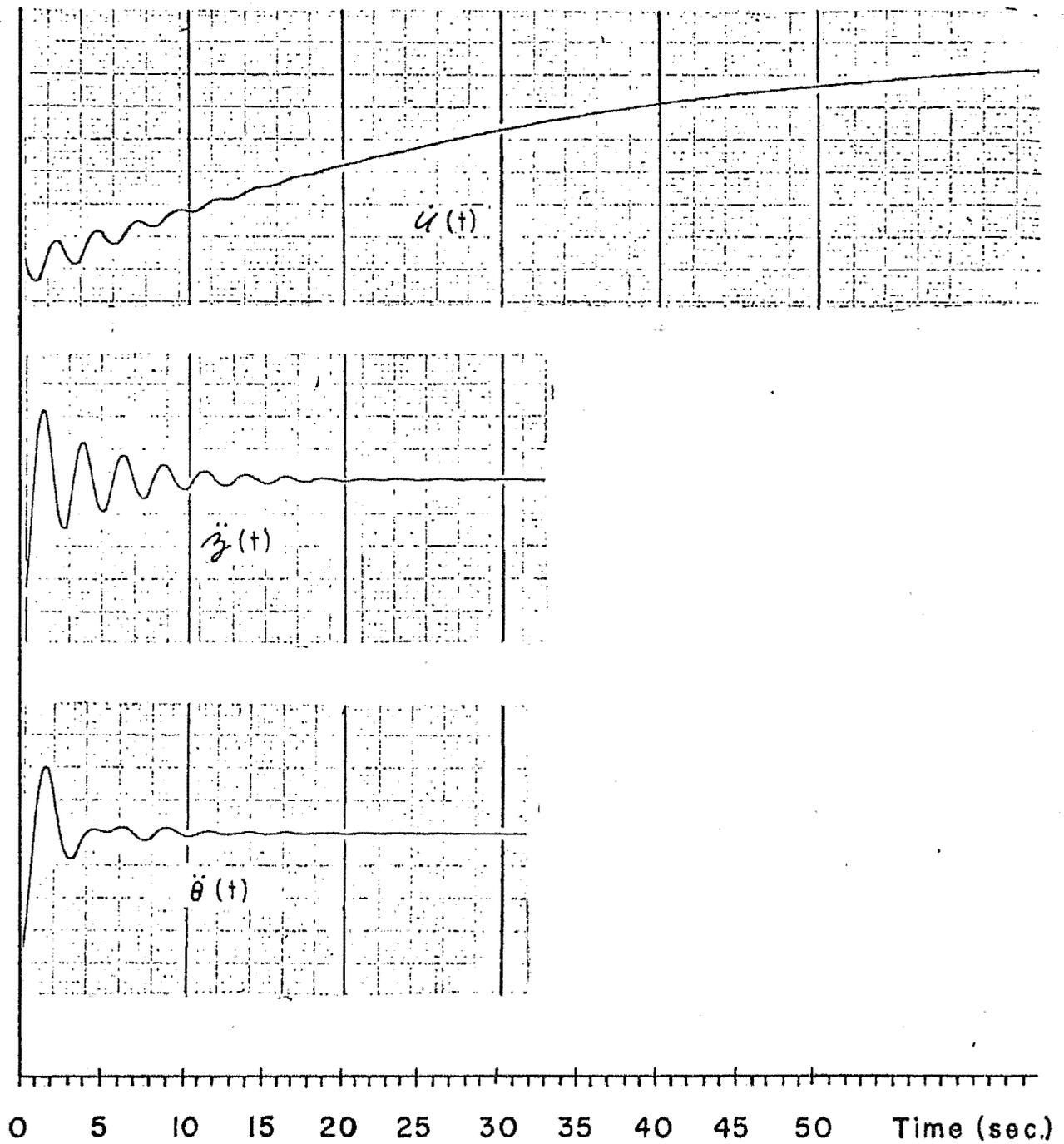


Figure 13. Simulator Output

	VALUE	COUPLED VALUE	COMPUTED VALUE
U_u	-0.033	-0.035	-0.035
Z_{γ}	-6.27	-6.51	-7.0
$Z_{i\gamma}$	-0.667	-0.411	-0.43
O_{β}	-3.67	-3.57	-3.38
O_{β}	-0.66	-0.914	-0.94

Figure 14. Computed Values of Derivatives