# MATCHING, SOCIAL WELFARE AND ORDINAL APPROXIMATION

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# ABSTRACT

Many important problems involve agents with preferences for different outcomes. Such settings include, for example, social choice and matching problems. Although the quality of an outcome to an agent may be measured by a numerical utility, it is often not possible to obtain these exact utilities when forming a solution. This can occur because eliciting numerical information from the agents may be too difficult, the agents may not want to reveal this information, or even because the agents themselves do not know the exact numerical values. On the other hand, eliciting *ordinal* information (i.e., the preference ordering of each agent over the outcomes) is often much more reasonable. Because of this, there has been a lot of recent work on *ordinal approximation algorithms*: these are algorithms which only use ordinal preference information as their input, and yet return a solution provably close to the optimal one. In other words, these are algorithms which only use limited ordinal information, and yet can compete in the quality of solution produced with omniscient algorithms which know the true (possibly latent) numerical utility information.

In this thesis, we study approximation algorithms for matching, social choice, facility assignment and other problems using agents' ordinal preferences and some other information in various settings. The basic assumption in this work is that the agents, facilities and candidates lie in a metric space, and the social welfare or cost depends on the distances among them in the metric.

First, we study ordinal approximation algorithms for maximum-weight bipartite matchings. We designed and analyzed mechanisms for three different levels of ordinal preferences: one-sided, two-sided and total ordering. We also consider settings where only the top preferences of the agents are known to us, instead of their full preference orderings. The results show that the approximation improves as more ordinal information is revealed.

Then we consider general facility location and social choice problems in the setting that besides ordinal preferences of the agents, the exact locations of the facilities/candidates are also given. Due to this extra information about the facilities, we are able to form powerful algorithms which have small *distortion*, i.e., perform almost as well as omniscient algorithms (which know the true numerical distances between agents and facilities) but use only ordinal information about agent preferences. We analyze many general problems including matching,

$k$-center, and $k$-median, and present black-box reductions from omniscient approximation algorithms with approximation factor $\beta$ to ordinal algorithms with approximation factor $1+2\beta$; doing this gives new ordinal algorithms for many important problems, and establishes a toolkit for analyzing such problems in the future.

Finally, we develop new voting mechanisms for social choice problems given voters' ordinal preferences as well as a small amount of information about the voters' preference *strengths*. We provide mechanisms with much better distortion when this extra information is known as compared to mechanisms which use only ordinal information. We quantify tradeoffs between the amount of information known about preference strengths and the achievable distortion. We further provide advice about which type of information about preference strengths seems to be the most useful.

# CHAPTER 1
# Introduction

Matching and assignment problems have been well studied in many applications, such as school choice [1],[2], house allocation [3],[4], facility assignment [5], stable marriage [6],[7] or stable roommates [8], social choice [9],[10] etc. These problems deal with a set of agents that have preferences over a set of facilities or agents. It is natural to assume that the agents prefer facilities with higher utilities (or lower cost). The goal is usually to design mechanisms to get a matching or assignment that maximize social welfare or minimize social cost, or to construct a stable matching/assignment.

In this work, we consider matching, social choice and other problems in a metric space, which means the distances between agents and facilities or candidates obey the triangle inequality. When the goal is to maximize the social welfare, the distances represent agents' utility, thus agents prefer farther facilities. While if the goal is to minimize the social cost, the distances represent agents' cost, and agents prefer closer facilities. There are different social cost objectives, e.g. to minimize the total social cost or to minimize the cost of the median/maximum agent, and various constraints such as facility capacities, opening fee, etc. For these objectives and constraints, if we know all the numerical distances in the metric space, we can obtain the optimal matching/assignment, although some problems might not have polynomial time solutions. However, agents might not be willing to give their numerical utilities/costs, or it might be difficult for them to realize the actual numbers. Instead, it is much easier to get the ordinal preferences of agents. For example, it is more natural to say"I prefer Hospital X to Hospital Y" than "Hospital X is 70 miles away and Hospital Y is 100 miles away from my home", and the former one also gives less private information. There has been ordinal approximation work in different settings [5],[11]–[21] to design mechanisms

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using only ordinal preferences to compare with the optimal solution given full numerical values.

In Chapter 2, we discuss related work in the matching, social choice and ordinal approximation problems. In Chapter 3, we study ordinal approximation algorithms for maximum-weight bipartite matchings given different levels of ordinal information. In Chapter 4, we study the general facility assignment problem and social choice problems, in the setting that we are given the agents' ordinal preferences, as well as the facilities' exact locations. In Chapter 5, we study the social choice problem with agents' ordinal preferences and a small amount of information about the voters' preference strengths. In Chapter 6, we discuss open problems and future directions.

## 1.1 Ordinal Approximation and Distortion in A Metric Space

As discussed above, the main goal of this thesis is to approximate the optimal solutions for matching, social choice and other problems using ordinal preferences. To qualify the performance of an ordinal algorithm to minimize social cost, we define approximation ratio as the worst-case ratio of its social cost to the social cost of the optimal algorithm which has access to the true underlying numerical information. This approximation ratio is also referred to as the *distortion* of a mechanism in social choice. Similarly, if the goal is to maximize social welfare instead of minimize social cost, the approximation ratio is defined as the worst-case ratio of the social welfare of the optimal solution to the social welfare of the oridinal algorithm.

### 1.1.1 Metric Space

As in some related researches [7],[18],[22], we assume that the agents, facilities and candidates are points in a metric space, and the weight of the edge connecting two nodes are the distances between them. For example, in facility assignment problems, it is quite straightforward to assume that agents and facilities are in a metric space, that minimizing social cost corresponds to minimizing the total distances in the assignment, that is, to find a minimum weight bipartite matching. In social choice, the distances between a voter and a candidate may represent the difference between their opinions, and voters prefer closer candidates. Another example is mixed doubles tennis players. Suppose each player's skill is denoted by a point in a metric space, and the distances between them represents the differences between their skills. Everyone prefer a partner with complementary skills, so the target is to find a maximum weight bipartite matching.

In a metric space, edges weights obey triangle inequality, that is, for $x, y, z \in \mathcal{N}$, $x(x, y) \leq w(x, z) + w(y, z)$. In bipartite graph, we assume that if agents lie in a metric space, then $\forall x_1, x_2 \in \mathcal{X}, \forall y_1, y_2 \in \mathcal{Y}, w(x_1, y_1) \leq w(x_1, y_2) + w(x_2, y_1) + w(x_2, y_2)$.

## 1.2 Tradeoffs Between Information and Ordinal Approximation for Bipartite Matching

Ordinal approximation is all about being able to produce good results with only limited information. Because of this, it is important to quantify how well algorithms can perform as more information is given. If the quality of solutions returned by ordinal algorithms greatly improves when they are provided more information, then it may be worthwhile to spend a lot of resources in order to acquire such more detailed information. If, on the other hand, the improvement is small, then such an acquisition of more detailed information would not be worth it. Thus the main question we consider in Chapter 3 is: *How does the quality of ordinal algorithms improve as the amount of information provided increases?*

In Chapter 3, we specifically consider this question in the context of computing a maximum-utility matching in a metric space. Matching problems, in which agents have preferences for which other agents they want to be matched with, are ubiquitous. The maximum-weight metric matching problem specifically provides solutions to important applications, such as forming diverse teams and matching in friendship networks (see [18],[19] for much more discussion of this). Formally, there exists a complete undirected bipartite graph for two sets of agents $\mathcal{X}$ and $\mathcal{Y}$ of size $N$, with an edge weight $w(x, y)$ representing how much utility $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ derive from their match; these edge weights satisfy the triangle inequality. The goal is to form a perfect matching between $\mathcal{X}$ and $\mathcal{Y}$, in order to approximate the maximum weight matching as much as possible using only the given ordinal information instead of actual numerical edge weights.

**Types of Ordinal Information** Ordinal approximation algorithms for maximum weight matching have been considered before in [18],[19], although only for complete graphs; algorithms for bipartite graphs require somewhat different techniques. Our main contribution, however, lies in considering many types of ordinal information, forming different algorithms for each, and quantifying how much better types of ordinal information improve the quality of the matching formed. Specifically, we consider the following types of ordinal information.

- The most restrictive model we consider is *one-sided preferences*. That is, only preferences for agents in $\mathcal{X}$ over agents in $\mathcal{Y}$ are given to our algorithm. These preferences are assumed to be consistent with the (hidden) agent utilities, i.e., if $x$ prefers $y_1$ to $y_2$, then it must be that $w(x, y_1) \geq w(x, y_2)$. Such one-sided preferences may occur, for example, when $\mathcal{X}$ represents people and $\mathcal{Y}$ represents houses. People have preferences over different houses, but houses do not have preferences over people. These types of preferences also apply to settings in which both sides have preferences, but we only have access to the preferences of $\mathcal{X}$, e.g., because the agents in $\mathcal{Y}$ are more secretive.

- The next level of ordinal information we consider is *two-sided preferences*, that is, both preferences for agents in $\mathcal{X}$ over $\mathcal{Y}$ and agents in $\mathcal{Y}$ over $\mathcal{X}$ are given. This setting could apply to the situation that two sets of people are collaborating, and they have preferences over each other, or of a matching between job applicants and possible employers. As we consider the model in a metric space, the distance (weight) between two people could represent the diversity of their skills, and a person prefers someone with most diverse skills from him/her in order to achieve the best results of collaboration.

- The most informative model which we consider in Chapter 3 is that of *total-order*. That is, the order of all the edges in the bipartite graph is given to us, instead of only local preferences for each agent. In this model, global ordinal information is available, compared to the preferences of each agent in the previous two models. Studying this setting quantifies how much efficiency is lost due to the fact that we only know ordinal information, as opposed to the fact that we only know *local* information given to us by each agent.

Comparing the results for the above three information types allows us to answer questions like: "Is it worth trying to obtain two-sided preference information or total order information when only given one-sided preferences?" However, above we always assumed that for an agent $x$, we are given their entire preferences for all the agents in $\mathcal{Y}$. Often, though, an agent would not give their preference ordering for all the agents they could match with, and instead would only give an ordered list of their top preferences. Because of this, in addition to the three models described above, we also consider the case of *partial* ordinal preferences, in which only the top $\alpha$ fraction of a preference list is given by each agent of $\mathcal{X}$. Thus for $\alpha = 0$ no information at all is given to us, and for $\alpha = 1$ the full preference ordering of an agent is given.

Considering partial preferences tells us when, if there is a cost to buying information, we might choose to buy only part of the ordinal preferences. We establish tradeoffs between the percentage of available preferences and the possible approximation ratio for all three models of information above, and thus quantify when a specific amount of ordinal information is enough to form a high-quality matching.

### 1.2.1 Our Contributions

We show that as we obtain more ordinal information about the agent preferences, we are able to form better approximations to the maximum-utility matching, even without knowing the true numerical edge weights. Our main results are shown in Figure 1.1 and Table 1.1.

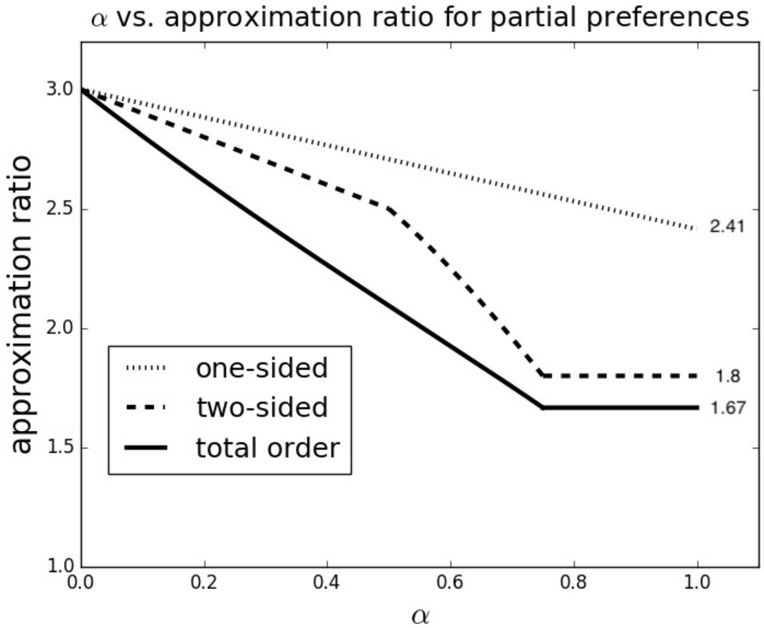


**Figure 1.1: $\alpha$ vs. approximation ratio for partial information. As we obtain more information about the agent preferences ($\alpha$ increases), we are able to form better approximation to the maximum-weight matching. The tradeoff for one-sided preferences is linear, while it is more complex for two-sided and total order.**

**Table 1.1: $\alpha$ vs. approximation ratio for partial information.**

| approximation ratio | $\alpha = 0$ | $0 < \alpha < 1$ | $\alpha = 1$ |
|---|---|---|---|
| one-sided | 3 | $(3 - (2 - \sqrt{2})\alpha)$ | 2.41 |
| two-sided | 3 | $((3 - 2\alpha)(3 - \alpha))/(2\alpha^2 - 3\alpha + 3)$ | 1.8 |
| total order | 3 | $(2 + \sqrt{1 - \alpha})/(2 - \sqrt{1 - \alpha})$ | 1.67 |

Using only one-sided preference information, with only the order of top $\alpha N$ preferences given for agents in $\mathcal{X}$, we are able to form a $(3 - (2 - \sqrt{2})\alpha)$-approximation. We do this by combining random serial dictatorship (RSD) with purely random matchings. When $\alpha = 1$, the algorithm yields a $(\sqrt{2} + 1)$-approximation. This is the first non-trivial analysis for the performance of *RSD* on maximum bipartite matching in a metric space, and this analysis is one of our main contributions.

Given two-sided information, with the order of top $\alpha N$ preferences for agents in both $\mathcal{X}$ and $\mathcal{Y}$, we can do significantly better. When $\alpha \geq \frac{1}{2}$, adopting an existing framework in [18], by mixing greedy and random algorithms, and adjusting it for bipartite graphs, we get a $\frac{(3-2\alpha)(3-\alpha)}{2\alpha^2-3\alpha+3}$-approximation. When $\alpha \leq \frac{1}{2}$, the framework would still work, but would not produce a good approximation. We instead design a different algorithm to get better results. Inspired by $RSD$, we take advantage of the information of preferences from both sets of agents, adjust $RSD$ to obtain "undominated" edges in each step, and finally combine it with random matchings to get a $(3-\alpha)$-approximation. When $\alpha \geq \frac{3}{4}$, the algorithm yields a 1.8-approximation.

For the total-ordering model, the order of top $\alpha N^2$ heaviest edges in the bipartite graph is given. We use the framework in [18] again to obtain a $\frac{2+\sqrt{1-\alpha}}{2-\sqrt{1-\alpha}}$-approximation. Here we must re-design the framework to deal with the cases that $\alpha \leq \frac{3}{4}N$, which is not a straight-forward adjustment. When $\alpha \geq \frac{3}{4}N$ the algorithm yields a $\frac{5}{3}$-approximation.

Finally, in Section 3.5 we analyze the case when edge weights cannot be too different: the highest weight edge is at most $\beta$ times the lowest weight edge in one-sided model. When the edge weights have this relationship, we can extend our analysis to give a $(\sqrt{\beta - \frac{3}{4}} + \frac{1}{2})$-approximation, even without assuming that edge weights form a metric.

## 1.3 Ordinal Approximation for Social Choice, Matching, and Facility Location Problems Given Candidate Positions

In Chapter 4 we consider general facility location problems, in which sets of agents $\mathcal{A}$ and facilities $\mathcal{F}$ are located in a metric space, and our goal is to assign agents to facilities (as well as choose which facilities to open) so that agents are assigned to facilities which are close to them. For example, $\mathcal{F}$ may consist of the possible locations to open new stores, and the goal may be that all agents have a store near them, or that the sum of agent distances to the stores they are assigned to is small, etc. This setting also captures many social choice problems, in which the facilities correspond to candidates, and the goal would be to choose a single candidate (and assign all agents to this candidate) so that the distances of the agents from the chosen candidate are small. Our setting also captures matching and many related problems, in which we would open all facilities, but are only able to assign one agent to each facility, thus forming a matching between agents and facilities; facilities here could correspond to houses or items, for example.

As mentioned before, we assume that only ordinal information about the distances between agents and facilities is known. However, although the locations and numerical preferences of the agents are usually difficult to obtain, the locations of facilities are mostly public information. The locations of political candidates in ideological space can be reasonably well estimated based on their voting records and public statements. When forming a survey about new stores to open, we may not know exactly how much the customers would prefer one store over the other since the customer locations may be private, but the locations of the possible stores themselves are public knowledge. The main difference between our work and previous work in this area is that we assume:

*While only ordinal information about agent preferences is known, we know the exact locations of the possible facilities $\mathcal{F}$.*

As we discuss below, this extra information about the locations of the facilities relative to each other allows us to produce much stronger algorithms, and show much nicer bounds on distortion. Note that this information might be available in the setting of other related papers, but not explicitly exploited. In fact, in many cases, we do not even need the full information about the locations of the facilities. The main message of Chapter 4 is that having a small amount of information about the candidates in social choice settings, or the facilities in facility location, allows us to obtain solutions which are provably *close to optimal* for a large class of problems even though the only information we have about the agent preferences is ordinal, and thus it is impossible (even given unlimited computational resources) to compute the *true* optimal solution.

### 1.3.1 Our Contributions

We begin by looking at the social choice setting, in which we have a set of $n$ agents $\mathcal{A}$ and $m$ candidates $\mathcal{F}$ in a metric space, and we are given an ordinal ranking of each agent for the candidates. This setting was considered in e.g., [12],[14]–[17],[23],[24]. In particular, for the objective of minimizing the total distance of the agents from the chosen candidate, [23] showed that Copeland and similar voting mechanisms always have distortion of at most 5. The best known deterministic algorithm is given by a recent paper [25] with a distortion of 4.236. While [23] showed that no deterministic voting mechanism can achieve a worst-case distortion of less than 3, it is still an open question to close the upper and lower bounds. In Chapter 4, we show that if we know the exact locations of the candidates in addition to the ordinal ranking of the agents, then there is a simple algorithm which achieves a distortion of

3, and no better bound is possible. In other words, while we do not know the true distances from agents to candidates, we can compute an outcome which is a 3-approximation *no matter what* the true distances are, as long as they are consistent with the ordinal preferences given to us. Moreover, this approximation is possible even if for each agent we are only given their favorite (i.e., top-choice) candidate: there is no need for the agents to submit a full preference ranking over all alternatives.

We also study other objective functions in addition to minimizing the total distance from agents to the chosen alternative. We give a natural deterministic voting mechanism which has distortion at most 3 for objectives such as minimizing the median voter cost, the egalitarian objective of minimizing the maximum voter cost, and many other objectives. This mechanism achieves all these approximation guarantees *simultaneously*, and moreover it does not need the exact locations of the candidates: it suffices to be given an ordinal ranking of the distances from each candidate to each other candidate. In other words, this mechanism is especially suitable for the case when candidates are a subset of voters, as our mechanism will obtain the ordinal ranking of each voter for all candidates, and this is the only information which would be required. Note that [23] proved that *no* deterministic mechanism can achieve a distortion better than 5 for the median objective; the reason why we are able to achieve a distortion of 3 here is precisely because we also know how each candidate ranks all the other candidates, in addition to how each voter ranks all candidates.

We then proceed to our general facility assignment model. We are given a set of agents and a set of facilities in a metric space. The distances between facilities are given, but the distances between agents and facilities are unknown; instead we only know ordinal preferences of the agents over the facilities which are consistent with the true underlying distances. There could be arbitrary constraints on the assignment, such as facility capacities, or constraints enforcing that some agents cannot be (or must be) assigned to the same facility, etc. A valid assignment is to assign each agent to a facility without violating the constraints. We consider many different social cost functions to optimize. For a general class of cost functions (essentially ones which are monotone and subadditive), we give a black-box reduction which converts an algorithm for the omniscient version of this problem (i.e., the version where the true distances are known) to an ordinal algorithm with small distortion. Specifically, if we have an omniscient algorithm which always produces an assignment which is a $\beta$-approximation of the optimum, then using it we can create an ordinal algorithm which only knows the ordinal preferences of the agents instead of their true distances to the facilities, but has distortion of at most $1 + 2\beta$.

**Table 1.2: Best known distortion of polynomial-time algorithms in different settings. “Omniscient” stands for the setting where all the distances between agents and facilities are known, and the numbers represent the best-known approximation ratios. The second column represent our setting, in which the ordinal preferences of the agents, and the numerical distances between facilities are known. The last column represents the pure ordinal setting in which only the agent ordinal preferences are known, but the distances between facilities are unknown; this setting has been previously studied, and we include the known lower bounds on the possible distortion in parentheses, including some which we prove in Chapter 4.**

| | Omniscient: full distances | Agents’ ordinal prefs and facility locations | Only agents’ ordinal prefs (lower bounds) |
|---|---|---|---|
| Total (Sum) Social Choice | 1 | 3 | 4.236(3) |
| Median Social Choice | 1 | 3 | 5(5) |
| Min Weight Bipartite Matching | 1 | 3 | $n$(3) |
| Egalitarian Bipartite Matching | 1 | 3 | -(2) |
| Facility Location | 1.488 [26] | 3.976 | $\infty$ ($\infty$) |
| $k$-center | 2 [27] | 5 | - (-) |
| $k$-median | 2.675 [28] | 6.35 | - ($\Omega(n)$) |

Many well-known problems fall into our facility assignment model; Table 1.2 summarizes some of our results. For example, classic facility location with facility costs, minimum weight bipartite matching, egalitarian bipartite matching, $k$-center, and $k$-median are all special cases. In particular our results show that if we are given unbounded computational resources, then it is always possible to form an assignment with distortion of at most 3 for these problems, and no better bound is possible simply due to the fact that we do not possess all the relevant information to compute the true optimum. This is a large improvement over previously known distortion bounds: for minimum cost ordinal matching the best-known distortion bound is $n$ using random serial dictatorship [5]; by using the knowledge of facility locations we are able to reduce this approximation ratio to 3.

## 1.4 Awareness of Voter Passion Greatly Improves the Distortion of Metric Social Choice

One often hears about 'where candidates stand' on issues, calling to mind a spatial model of preferences in social choice [29]–[34]. In proximity-based spatial models, voters' preferences over candidates are derived from their distances to each of the candidates in some issue space. In particular, we consider voters and candidates which lie in an arbitrary unknown metric space. Our work follows a recent line of research in social choice which considers this setting [12],[14]–[17],[23],[24],[35]–[40]. The distance between each voter and the winning candidate is interpreted as the cost to that voter. Naturally, one of the main goals is to select the candidate which minimizes the total Social Cost, i.e., the sum of costs of the voters.

The fundamental assumption and motivation in the previous related work is that the *strength* or intensity of voter preferences is not possible to obtain, and thus we must do the best we can with only ordinal preferences. And indeed, knowing the exact strength of voter preferences is usually impossible. In many settings, however, *some* cardinal information about the ardor of voter preferences is readily available or obtainable, and is often used to affect outcomes and make better collective decisions. For example, a decision in a meeting may be decided in favor of a minority position if those in the minority are significantly more adamant or passionate about the issue than the apathetic majority, as revealed during discussion or debate. In political campaigns, the amounts of monetary donations, activists attending rallies, and other measures of "grass-root support" can cause a candidate to become a de-facto front-runner even before an official election or primary is ever held. Because of this, in Chapter 5 we ask the question: "How much can the quality of selected candidates be improved if we know some *small* amount of information about the *strength* of voter preferences?"

There are many different approaches for modeling, measuring, eliciting, and aggregating the strength or intensity of voter preferences [41],[42]. Such measures can be done through survey techniques, measuring the total amount of monetary contributions, amounts of excitement and time people spend volunteering or advocating for particular issues, etc. All such measures are by their very nature imprecise. And yet while it is unreasonable to assume that exact strength of preference is known for every voter, it is certainly possible to obtain insights such as "there are many more voters who are passionate about candidate A as compared to candidate B", or quantify the approximate amount of extreme preference strengths as opposed to the voters who are mostly indifferent. As we show in Chapter 5, even such a small amount of information about aggregate preference strengths or the amount of passionate voters can greatly improve distortion, and allow mechanisms which provably result in outcomes that are close to optimal. In fact, knowing only a single additional bit of information for each voter (i.e., do they prefer A to B strongly, or not strongly?) is enough to greatly improve distortion.

### 1.4.1 Our Contributions

What type of knowledge of the strengths of voter preferences is most useful and advantageous? What voting mechanisms should be used in order to minimize distortion if you have access to more information than only ordinal preferences? If you could gather data about voter preferences in different ways, what should you aim for in order to reduce distortion? These are some of the questions which we attempt to illuminate in Chapter 5.

In this work, we study the possible distortion with different levels of voter preference strength information. A summary of our results is shown in Table 1.3. We begin with the setting in which we are given the voters' ordinal preferences, as well as a threshold $\tau \geq 1$ of voter preference strength. In other words, for any two candidates $P$ and $Q$, we know the number of voters who prefer $P$ to $Q$, as well as how many of them prefer $P$ to $Q$ by at least a factor of $\tau$ (i.e., $d(i, P) < \frac{1}{\tau} d(i, Q)$). Based on only this information about the voter preferences (and the fact that the voters and candidates are embedded in some arbitrary unknown metric space), we are able to provide new voting mechanisms with much better distortion than possible when only knowing ordinal preferences. For the case that there are only two candidates, we provide a mechanism which achieves provably the best possible distortion of $\max\{\frac{\tau+2}{\tau}, \frac{3\tau-1}{\tau+1}\}$, as shown in Figure 1.2. For the setting with more than two candidates, we get a distortion of $\min\{\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\} + 2, \max\{(\frac{3\tau-1}{\tau+1})^2, (\frac{\tau+2}{\tau})^2\}\}$ as shown in Figure 1.3. Note that when $\tau = 1$, we get a distortion of 5. A recent paper [25] shows a deterministic algorithm that gives a distortion of 4.236.

**Table 1.3: Distortion in different settings.**

| Distortion | Two Candidates | More than Two Candidates |
|---|---|---|
| Preferences and a threshold $\tau$ | $\max\{\frac{\tau+2}{\tau}, \frac{3\tau-1}{\tau+1}\}$ | $\min\{\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\}+2, \max\{(\frac{3\tau-1}{\tau+1})^2, (\frac{\tau+2}{\tau})^2\}\}$ |
| $m$ thresholds $\tau_1, \ldots, \tau_m$ | $\max_{1\le l\le m}\{\frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1}\}$ | $\max_{1\le l\le m}\{(\frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1})^2\}$ |
| Exact preference strengths | $\sqrt{2}$ | 2 |

From Figures 1.2 and 1.3, we can see that the distortion is minimized when $\tau = 1+\sqrt{2}$ in both settings. With only voter preferences being known, the best known deterministic distortion bounds are 3 for two candidates [23], and 4.236 for multiple candidates [25]. Interestingly, if we are also allowed to choose a threshold $\tau$, our results indicate that the optimal thing to do is to differentiate between candidates with a lot of supporters who prefer them at least $1+\sqrt{2}$ times to other candidates, and candidates which have few such supporters. By obtaining this information, we can improve the quality of the chosen candidate from a 3-approximation to only a 1.83 approximation (for 2 candidates), and from a 4.236-approximation to a 3.35-approximation (for $\geq 3$ candidates). This is a huge improvement obtained with relatively little extra cost in information gathering.

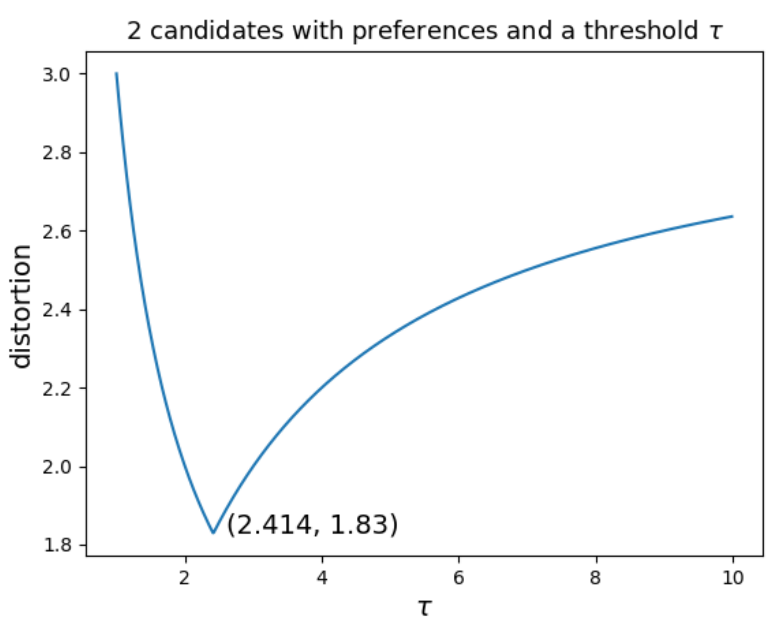


**Figure 1.2: Distortion for two candidates with preferences and a threshold $\tau$.**

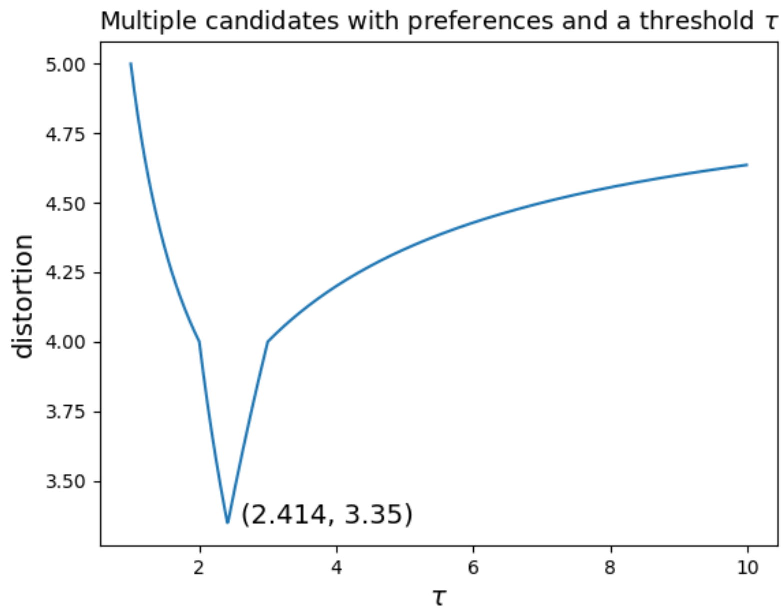


**Figure 1.3: Distortion for multiple candidates with preferences and a threshold $\tau$.**

In Section 5.3 we consider the case when we only know the preferences of voters who feel strongly about their choice (prefer $P$ to $Q$ by at least $\tau$ times), but do not know the preferences of voters who are relatively indifferent. We show that knowing how many voters feel strongly about a candidate is actually *more* important than knowing the ordinal preferences of all voters when attempting to minimize distortion: for example if we have $\tau = 2$ we can obtain a distortion of 2 as well, even if we don't know the preferences of all voters.

We then consider a more general case in Section 5.4. Suppose we have $m$ different thresholds $\{1 \leq \tau_1 < \tau_2 < \ldots < \tau_m\}$, and voters report the largest threshold which their preference strength exceeds for each pair of candidates. As $m$ gets larger, the information about preference strengths gets less coarse; for most settings it would be realistic to assume that $m$ is small, but we provide a result which is as general as possible. With this information, we give a mechanism achieving the provably the best distortion of $\max_{1 \leq l \leq m}\{\frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1}\}$ in the two candidates setting, and a distortion of $\max_{1 \leq l \leq m}\{(\frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1})^2\}$ in the multiple candidates setting. Note that knowing all the preference strengths *exactly* is still not enough to always be able to choose the optimal candidate: the preference strengths are relative ("I like A twice as much as B") as opposed to absolute. We never obtain information about how the costs of different voters compare to each other, the only thing we know is that the voters lie in a metric space. In fact, when we know the exact preference strengths of every voter, we obtain a distortion bound of $\sqrt{2}$ in the two candidates setting, and a distortion of 2 in the multiple candidates setting. Moreover, we prove that even knowing the exact preference strengths, it is not possible to obtain distortion better than $\sqrt{2}$ in the worst case.

# CHAPTER 2
# Background and Related Work

Ordinal approximation [43] for the minimum social cost (or maximum social welfare) with underlying utilities/distances between agents and alternatives has been studied in many settings including social choice [11]–[17],[44],[45], matchings [5],[18]–[20],[22],[46],[47], secretary problems [48], participatory budgeting [49], general graph problems [18],[21] and many other models in recent years. The general assumption of the ordinal setting is that we only have the ordinal preferences of agents over alternatives, and the goal is to form a solution that has close to optimal social cost. There are different models: social choice, matching, facility location, etc.; different objectives: minimizing social cost, maximizing social welfare, total cost objective, median objective, egalitarian objective, etc.; different assumptions on utility or cost functions: unit-sum, unit-range, metric space, etc. We survey related previous work on matching, facility assignment and social choice in this section.

## 2.1 Matching and Facility Assignment Problems

Beside the assumption in this thesis that the underlying utility/cost exists while we only know the ordinal preferences, there are several other reasons that we are only able to get approximations of optimal solutions in some situations. For example, there are studies on linear or near linear time for maximum weight matching, [50] gets a $(\frac{2}{3}-\epsilon)$ approximation and [51] guarantees a $(\frac{3}{4}-\epsilon)$ approximation. The main concern is to get efficient running time, which is important when dealing with large graphs. Also, there are researches on truthful algorithms [52],[53], using only ordinal preferences to guarantee that no agent intend to lie about his/her utility to get a better payoff with cost of the social welfare. We discuss related work on matching and facility assignment problems in this section.

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### 2.1.1 Ordinal Matching Mechanisms

The maximum/minimum weight perfect matching problems could be solved by the Hungarian method [54]. But the algorithm is not truthful as agents could lie about their utilities to get a better payoff. As a result, researchers often choose truthful algorithms that gives approximations of the optimal solution. Top trading cycle (TTC) [55] is a truthful and Pareto optimal algorithm by keep searching for cycles induced by agent's top choices and make assignment in the cycle until all agents and items are matched. Serial Dictatorship (SD) and Random Serial Dictatorship (RSD, also called Random Priority) [3] are also truthful and Pareto optimal. The process of SD is very straightforward: select an arbitrary agent, assign his/her top choice, until all agents are assigned. RSD is a randomized version of SD that select agent uniformly at random in each step. Probabilistic Serial(PS) [56] is a weakly truthful algorithm that for each item, give a fraction of it to every agents prefer it, and repeat this until all items are fully consumed.

### 2.1.2 Assumptions of Numerical Utilities in Ordinal Matchings

When we consider matching using only ordinal information, if the underlying weights are arbitrary, the approximation ratio could be unbounded. For example, suppose there are two agents and two items. We only know the ordinal preferences of agents without numerical values, and our goal is to find a matching that maximize total utilities. There are only two possible perfect matchings: $\{(agent1, item1), (agent2, item2)\}$ and $\{(agent1, item2), (agent2, item1)\}$. If both agents prefer $item1$ to $item2$, there is no way for us to distinguish between the two agents, and decide which assignment induce a better social welfare. Note that the difference between total utilities generated by these two assignments could be arbitrarily huge, so even if we use randomized algorithms, it is still not possible to bound the expected approximation ratio. Therefore, it is a common approach to use normalized valuation functions, such as unit-sum (each agent has a total utility of 1 for all the items) and unit-range (for any agent, the maximum utility to any item is 1 and the minimum utility to any item is 0) [47]. Linear welfare factor [46] assumes that an agent's utility linearly decreases in his/her preference list. There are also assumptions that all the agents are points in a metric space, which also constrain the underlying weights, mainly by triangle inequality.

### 2.1.3 Ordinal Approximation in Matching Problems

Previous work on forming good matchings can largely be classified into the following classes. First, there is a large body of work assuming that numerical weights or utilities don't exist, only ordinal preferences. Such work studies many possible objectives, such as forming stable matchings (see e.g., [57],[58]), or maximizing objectives determined only by the ordinal preferences (e.g., [52],[59]). Second, there is work assuming that numerical utilities or weights exist, and are *known* to the matching designer. Unlike the above two settings, we consider the case when numerical weights *exist*, but are latent or *unknown*, and yet the goal is to approximate the true social welfare, i.e., maximum weight of a perfect matching in Chapter 3. Note that although some previous work assumes that all numerical utilities are known, they often still use algorithms which only require ordinal information, and thus fit into our framework; we discuss some of these results below.

Similar to our one-sided model in Chapter 3, house allocation [3] is a popular model of assigning $n$ agents to $n$ items. Bhalgat et al. [46] studied the ordinal welfare factor and the linear welfare factor of RSD and other ordinal algorithms. Krysta et al. [4] studied both maximum matching and maximum vertex weight matching using an extended RSD algorithm. These either used objectives depending only on ordinal preferences, such as the size of the matching formed, or used node weights (as opposed to edge weights). Filos-Ratsikas et al. [22] and Christodoulou et al. [47] assumed the presence of numerical agent utilities and studied the properties of RSD. Crucially, this work assumed normalized agent utilities, such as unit-sum or unit-range. This allowed [22],[47] to prove approximation ratios of $\Theta(\sqrt{n})$ for RSD. Instead of assuming that agent utilities are normalized, we consider agents in a metric space; this different correlation between agent utilities allows us to prove much stronger results, including a constant approximation ratio for RSD. Kalyanasundaram et al. studied serial dictatorship (SD) for maximum weight matching in a metric space [60], and gave a 3-approximation for SD in this, while we are able to get a tighter bound of 2.41-approximation.[1]

Besides maximizing social welfare, minimizing the social cost of a matching is also popular. [5] studied the approximation ratio of RSD and augmentation of serial dictatorship (SD) for minimum weight matching in a metric space. Their setting is very similar to our

[1]Note that many of the papers mentioned here specifically attempt to form *truthful* algorithms. While RSD is certainly truthful, in Chapter 3 we attempt to quantify what can be done using ordinal information in the presence of latent numerical utilities, and leave questions of truthfulness to future work.

work in Chapter 3, except that we consider the maximization problem, which has different applications [18],[19], and allows for a much better approximation factor (constant instead of linear in $n$) using different techniques.

The work most related to our work in Chapter 3 is [18],[19]. We use an existing framework [18] for the two-sided and the total-order model. While the goal is the same: to approximate the maximum weight matching using ordinal information, our work in Chapter 3 is different from [18] in several aspects. [18] only considered approximating the true maximum weight matching for non-bipartite complete graphs. We instead focus on bipartite graphs, and especially on considering different levels of ordinal information by analyzing three models with increasing amount of information, and also consider partial preferences. Although we use similar techniques for parts of two-sided and total-order model analysis, they need significant adjustments to deal with bipartite graphs and partial preferences; moreover, the method used for analyzing the one-sided model is quite different from [18].

### 2.1.4 House Allocation Problems

House allocation [3] is a model of assigning $n$ agents to $n$ items, given that each agent has preferences over the items, and it is possible some items are not acceptable for an agent. This model is also referred to as one-sided matching problem, the goal is to form a matching to maximize the number of matched agents or total utilities, or minimize total cost. Represent the agents and items in a bipartite graph, with utility as weight of edges between them, these problems turn into maximum cardinality or maximum/minimum weight matching problems in the graph.

Bhalgat et al. [46] define ordinal welfare factor as number of agents that are at least as happy as in any matching, and shows that both RSD and PS gives a $\frac{1}{2}$ approximation. In linear welfare factor assumption, RSD gives [0.526, $\frac{2}{3}$], PS gives roughly $\frac{2}{3}$.

Filos-Ratsikas et al. [22] show that the approximation ratio of RSD for unit-sum and unit-range valuation functions both have $\Theta(n^{-\frac{1}{2}})$ approximation, and RSD is also the best truthful (in expectation) and ordinal algorithm in the problem setting.

Christodoulou et al. [47] show that both RSD and PS have the price of anarchy $O(\sqrt{n})$ considering unit-sum and unit-range valuation functions. They also show the lower bound of price of Anarchy of is $\Omega(\sqrt{n})$, and the lower bound of price of anarchy for any deterministic mechanism is $\Omega(n^2)$.

Krysta et al. [4] studied both maximum matching and maximum vertex weight matching using an extended RSD algorithm, and get $\frac{e}{e-1}$ approximation in both cases. Note that maximum (number) matching is a special case of maximum weight matching that all edge weights are set to one, and maximum vertex weight is another special case that all the edge connected to the same agent have the same weight [61].

Adamczyk et al. [53] shows that in dichotomous (binary) setting, there exist a truthful and symmetric algorithm that gives a 0.69-approximation (1.45) to the maximum social welfare (matching). RSD in dichotomous setting gives a 3-approximation. In normalized setting with $n$ agents and $n$ items, let $\nu(\mathcal{O})$ denote the maximum social welfare, RSD returns a matching of expected social welfare at least $\frac{1}{e}\frac{\nu(\mathcal{O})^2}{n}$, and no truthful algorithm can give a social welfare better than $\frac{\nu(\mathcal{O})^2}{n}$.

### 2.1.5 Facility Assignment Problems

Facility assignment problems [62],[63] usually contain different numbers of agents and facilities, and each facility has a finite capacity. The target is to maximize social welfare or minimize social cost. House allocation problem is actually a special case of assignment problems that has same number of agents and facilities, and every facility has a capacity of 1. Pentico [64] gave a through survey of different types of assignment problems.

Caragiannis et al. [5] study the approximation ratio of RSD and augmentation of serial dictatorship (SD) for minimum weight matching (minimum social cost) in metric space. If the capacities of facilities are augmented by any integer g, SD has an approximation ratio $\frac{g}{(g-2)}$ when $g \geq 3$. Moreover, the approximation ratio of SD and RSD are bounded by: $ratio(SD, g = 1) \geq 2^n - 1$, $ratio(SD, g = 2) >= \log{(n + 1)}$, $ratio(SD, g \geq 3) \geq \frac{g}{(g-2)} - \delta$, for any $\delta > 0$. $ratio(RSD, g = 1) \geq n^{0.26}$ .

Cechlárová et al. [65] study assignment problems that agents have strict preferences over objects, but the preference list could be incomplete. The paper discussed about whether an agent can obtain or necessarily obtains a given object under SD. Both problems are hard if agents have preference lists of length at most 3, but have linear algorithms for length at most 2. They also showed the lower bound of price of anarchy for any deterministic mechanism is $\Omega(n^2)$.

### 2.1.6 Online Matching

Online bipartite matching problems assume the agents arrive in a order, and immediately matched to an item once arrived. This is another type of incomplete information that make it not possible to achieve the optimal solution, and approximation approaches are widely studied. For maximum cardinality matching, only ordinal information is considered [66]. While for maximum weight matching, numerical utility values are used in some studies [61],[67]. Kalyanasundaram et al. studies serial dictatorship for both minimum and maximum weight matching in a metric space [60]. Epstein et al. [68] studies online preemptive matching, which means the agents can regret previous decisions remove matched edges.

## 2.2 Social Choice

Computational social choice [9],[10] is a computer science perspective for the design and analysis of strategies to make decisions based on preferences of multiple agents. One application of social choice is to aggregate agents' preferences into group preferences. Another example is voting: a set of agents have ordinal preferences over a set of alternatives (candidates), and only one winner will be chosen from the candidates. Generally speaking, stable matching, resource allocation and facility assignment problems also fall in the area of social choice, as they all concern about making decisions based on preferences of agents. In this section, we focus on studies of agents' preference aggregation and voting mechanisms.

There are some desirable properties for preference aggregation, for example, *Pareto-optimality*, *independence of irrelevant alternatives (IIA)*, and *non-dictatorship*. Preference aggregation methods are described by social welfare functions. *Pareto-optimality* requires that if every agent prefer candidate $X$ to $Y$, then the whole group prefers $X$ to $Y$. *Independence of irrelevant alternatives* means that the group preferences on any two candidates only depends on agents' relative preferences on these two candidates (and not affected by any other candidates). A social welfare function is *Non-dictatorship* if there is not a single agent that can decide the group preference, regardless of the preferences of other agents. However, by Arrow's impossibility theorem [69], there is no social welfare function that satisfy *Pareto-optimality*, *IIA* and *non-dictatorship* when the number of candidates is at least three. If we only concern about the winner instead of the whole ranking list of the group preferences, the mechanism is represented by a social choice function. A variant of Arrow's impossibility theorem [70] states that there is no social choice function that satisfy *Pareto-optimality*, *IIA* and *non-dictatorship* and *weak axiom of revealed preference (WARP)*. Therefore, one research area of social choice is to relax at least one of the properties in Arrow's impossibility theorem to design relatively desirable preference aggregation or social choice functions [71]–[73].

There are many voting rules [9] to decide a single winner based on agents' ordinal preferences, such as *Borda's rule*, the *plurality rule*, the *anti-plurality rule*, *Copeland's rule*, the *maximin rule*, the *Ranked pairs rule*, the *STV rule* etc. Similar to Arrow's impossibility theorem, the Gibbard-Satterthwaite theorem [74],[75] states that if a voting rule is *non-imposing* (for every candidate, there exist some preference profiles that he/she is the winner) and *strategy-proof*, then it must be *dictatorial*. An assumption of the Gibbard-Satterthwaite theorem is that the preference profiles could be arbitrary, and some studies focus on restricted domains, such as single-peaked preferences [76],[77] that there always exist a Condorcet winner chosen by a strategy-proof social choice function.

### 2.2.1 Distortion in Social Choice Mechanisms

Our work in this thesis falls in the utilitarian approach of social choice [13],[78], which assumes that the preferences of agents are induced by underlying utility values over candidates. The distortion of social choice functions was first introduced in [44], to describe the ratio between the total utility of the optimal candidate and the candidate selected by a mechanism using only the ordinal preferences. Since then, two main approaches have emerged for analyzing the distortion of various voting mechanisms. One is assuming that the underlying unknown utilities or costs are normalized in some way, e.g., [13],[45],[49],[79]–[84]. Especially, Amanatidis et al. [85] study distortion with queries of voters' preference strength, which is similar to our model in Chapter 5, but with unit-sum or unrestricted utility functions. The second approach, which we take here, assumes all voters and candidates are points in a metric space [12],[14]–[17],[23],[24],[35]–[40],[86]. In particular, when the latent numerical costs that induce voter preferences over a set of candidates obey the triangle inequality, it is known that simple deterministic voting rules yield distortion which is always at most a small constant (5 for the well-known Copeland mechanism [23], and recently 4.236 for a more sophisticated, yet elegant, mechanism [25]). For other well-known mechanisms, there is a bound of $O(\ln m)$ for Single Transferable Vote (STV) [16]. In addition, [23] proved that no deterministic mechanism can have worst-case distortion better than 3, and [16] showed that all scoring rules for $m$-candidates have a distortion of at least $1 + 2\sqrt{\ln m - 1}$. Amanatidis et al. [85] studied preference strength with unit-sum or unrestricted utility functions. Goel et al. [14] showed that Ranked Pairs, and the Schulze rule have a worst-case distortion of at least 5, and the expected worst-case distortion of any (weighted)-tournament rule is at least 3. They also introduced the notion of "fairness" of social choice rules, discussed the fairness ratio of Copeland, Randomized Dictatorship, a general class of cost functions, and studied the relationship between fairness and distortion [86]. In Chapter 4, we show that if we know the exact locations of the candidates in addition to the ordinal ranking of the agents, then there is a simple algorithm which achieves a distortion of 3, and no better bound is possible.

While the above work, as well as our work in Chapter 4 and Chapter 5, only focuses on deterministic algorithms, the distortion of randomized algorithms in social choice has also been considered, see for example [12],[15],[24],[87]. In a slightly different flavor of result, [17],[37] consider the special case where candidates are randomly and independently drawn from the set of voters. While we leave the analysis of randomized algorithms which know the location of the facilities to future work, and consider the worst-case candidate locations, it is worth pointing out that our *deterministic* algorithm achieves a distortion of 3, which is also the best known distortion bound for any *randomized* mechanism which only knows the ordinal preferences of the agents. Similarly, another common goal is to form *truthful* mechanisms with small distortion for matching and social choice, as in [5],[15],[19]; we focus on general mechanisms in Chapter 4 in order to understand the limitations of knowing only certain kinds of ordinal information, and leave the goal of forming truthful mechanisms for future work.

For the median objective of social choice problems, [23] showed that Copeland gives a distortion of at most 5, while *no* deterministic mechanism can achieve a distortion better than 5. [12] also gave a randomized algorithm that has a distortion of at most 4. In Chapter 4, we are able to improve this bound to a tight worst-case distortion of 3 by a deterministic mechanism, because we also know how each candidate ranks all the other candidates, in addition to how each voter ranks all candidates.

#### 2.2.1.1 Randomized vs Deterministic Mechanisms

We restrict our attention to deterministic social choice rules, instead of randomized ones as in e.g., [12],[15],[24],[45], for several reasons. First, consider looking at our mechanisms from a social choice perspective, i.e., as voting rules that need to be adopted by organizations and used in practice. People are far more resistant to adopting randomized voting protocols.

This is because an election with a non-trivial probability of producing a terrible outcome is usually considered undesirable, even if the *expected* outcomes are good. There are many exceptions to this, of course, but nevertheless deterministic mechanisms are easier to convince people to adopt. Second, consider looking at our mechanisms from the point of view of approximation algorithms, i.e., as algorithms which attempt to produce an approximately-optimal solution given a limited amount of information. For traditional randomized approximation algorithms with guarantees on the quality of the expected outcome it is possible to run the algorithm several times, take the best of the results, and be relatively sure that you have achieved an outcome close to the expectation. In this setting of limited information, however, we cannot know the "true" cost of a candidate even after a randomized mechanism chooses it, and thus cannot take the best outcome after several runs. Therefore, unless stronger approximation guarantees are given than simply bounds on the expectation, it is quite likely that the outcome of a randomized algorithm in our setting would be far from the expected value. While randomized algorithms are certainly worthy of study even in our setting, and many interesting questions about them exist, we choose to focus only on deterministic algorithms.

#### 2.2.1.2 Preference Strength

Attempts to exploit preference strength information have led to various approaches for modeling, eliciting, measuring, and aggregating people's preference intensities in a variety of fields, including Likert scales, semantic differential scales, sliders, constant sum paired comparisons, graded pair comparisons, response times, willingness to pay, vote buying, and many others (see [41],[42],[88] for summaries). In our work we specifically consider only a small amount of coarse information about preference strengths, since obtaining detailed information is extremely difficult. Intuitively, any rule used to aggregate preference strengths must ask under what circumstances an 'apathetic majority' should win over a more passionate minority [89], and we provide a partial answer to this question when the objective is to minimize distortion.

Perhaps most related to our work in Chapter 5 is that of [12] which introduced the concept of *decisiveness*. Using our notation, [12] proves bounds on distortion under the assumption that *every* voter has a preference strength at least $\alpha$ between their top and second-favorite candidates. We, on the other hand, do not require that voters have any specific preference strength between any of their alternatives, and provide general mechanisms and distortion bounds based on knowing a bit more about voters (arbitrary) preference strengths. In other words, while [12] limits the possible space of voter preferences and locations in the metric space, we instead allow those to be completely arbitrary, but assume that we are given slightly more information about them.

In our model, when voter preference strength is less than the smallest threshold, they effectively abstain because their preferred candidate is unknown, and so any reasonable weighted majority rule must assign them a weight of 0. Therefore, our work also bears resemblance to literature on voter abstentions in spatial voting (see [39] and references therein). While there are major technical differences in our model and that of [39], at a high level the model of [39] is similar to a special case of ours with only two candidates and a single threshold on preference strengths (and no knowledge of voter preferences otherwise), which we analyze in Section 5.3.

# CHAPTER 3
# Tradeoffs Between Information and Ordinal Approximation for Bipartite Matching

In this chapter we study ordinal approximation algorithms for maximum-weight bipartite matchings. Such algorithms only know the ordinal preferences of the agents/nodes in the graph for their preferred matches, but must compete with fully omniscient algorithms which know the true numerical edge weights (utilities). Ordinal approximation is all about being able to produce good results with only limited information. Because of this, one important question is how much better the algorithms can be as the amount of information increases. To address this question for forming high-utility matchings between agents in $\mathcal{X}$ and $\mathcal{Y}$, we consider three ordinal information types: when we know the preference order of only nodes in $\mathcal{X}$ for nodes in $\mathcal{Y}$, when we know the preferences of both $\mathcal{X}$ and $\mathcal{Y}$, and when we know the total order of the edge weights in the entire graph, although not the weights themselves. We also consider settings where only the top preferences of the agents are known to us, instead of their full preference orderings. We design new ordinal approximation algorithms for each of these settings, and quantify how well such algorithms perform as the amount of information given to them increases.

## 3.1 Model and Notation

For all the problems studied in this chapter, we are given as input two sets of agents $\mathcal{X}$ and $\mathcal{Y}$ with $|\mathcal{X}| = |\mathcal{Y}| = N$. $G = (\mathcal{X}, \mathcal{Y}, E)$ is an undirected complete bipartite graph with weights on the edges. We assume that the agent preferences are derived from a set of underlying hidden edge weights $w(x, y)$ for each edge $(x, y)$, $x \in \mathcal{X}, y \in \mathcal{Y}$. $w(x, y)$ represents the utility of the match between $x$ and $y$, so if $x$ prefers $y_1$ to $y_2$, then it must be that $w(x, y_1) \geq w(x, y_2)$. Let $OPT(G)$ denote the complete bipartite matching that gives the maximum total edge weights. $w(G)$ of any bipartite graph $G$ is the total edge weight of the graph, and $w(M)$ of any matching $M$ is the total weight of edges in the matching.

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The agents lie in a metric space, by which we will only mean that, $\forall x_1, x_2 \in \mathcal{X}, \forall y_1, y_2 \in \mathcal{Y}, w(x_1, y_1) \leq w(x_1, y_2) + w(x_2, y_1) + w(x_2, y_2)$. We assume this property in all sections except for Section 3.5.

For the setting of one-sided preferences, $\forall x \in \mathcal{X}$, we are given a strict preference ordering $P_x$ over the agents in $\mathcal{Y}$. When dealing with partial preferences, only top $\alpha N$ agents in $P_x$ are given to us in order. We assume $\alpha N$ is an integer, $\alpha \in [0, 1]$. Of course, when $\alpha = 0$, nothing can be done except to form a completely random matching. For two-sided partial preferences, we are given both the top $\alpha$ fraction of preferences $P_x$ of agents $x$ in $\mathcal{X}$ over those in $\mathcal{Y}$, and vice versa. For the total order setting, we are given the order of the highest-weight $\alpha N^2$ edges in the complete bipartite graph $G = (\mathcal{X}, \mathcal{Y}, E)$.

## 3.2 One-sided Ordinal Preferences

For one-sided preferences, our problem becomes essentially a house allocation problem to maximize social welfare, see e.g., [4],[22],[47]. Before we proceed, it is useful to establish a baseline for what approximation factor is reasonable. Simply picking a matching uniformly at random immediately results in a 3-approximation (see Theorem 3.2.5), and there are examples showing that this bound is tight. Other well-known algorithms, such as Top Trading Cycle, also cannot produce better than a 3-approximation to the maximum weight matching for our setting. Serial Dictatorship, which uses only one-sided ordinal information, is also known to give a 3-approximation to the maximum weight matching for our problem [60]. Serial Dictatorship simply takes an arbitrary agent from $x \in \mathcal{X}$, assigns it $x$'s favorite unallocated agent from $\mathcal{Y}$, and repeats. Note that [60] used a greedy algorithm for the online maximum weight matching problem, and the algorithm is actually SD because the arbitrary arriving order in online problems describes how we pick agents in an arbitrary order. Unfortunately, it is not difficult to show that this bound of 3 is tight. Our first major result in this chapter is to prove that *Random* Serial Dictatorship always gives a $(\sqrt{2}+1)$-approximation in expectation, no matter what the true numerical weights are, thus giving a significant improvement to all the algorithms mentioned above.

**Algorithm 1:** Random Serial Dictatorship for Perfect Matching of one-sided ordering.

Initialize $M = \emptyset$, $G = (\mathcal{X}, \mathcal{Y}, E)$ ;
**while** $E \neq \emptyset$ **do**
    Pick an agent $x$ uniformly at random from $\mathcal{X}$ ;
    Let $y$ denote $x$'s most preferred agent in $\mathcal{Y}$ ;
    Take $e = (x, y)$ from $E$ and add it to $M$ ;
    Remove $x$, $y$, and all edges containing $x$ or $y$ from the graph $G$ ;
**end**
**Final Output:** Return $M$.

**Theorem 3.2.1.** *Suppose $G = (\mathcal{X}, \mathcal{Y}, E)$ is a complete bipartite graph on the set of nodes $\mathcal{X}, \mathcal{Y}$ with $|\mathcal{X}| = |\mathcal{Y}| = N$. Then, the expected weight of the perfect matching $M$ returned by Algorithm 1 is $\mathbb{E}[w(M)] \geq \frac{1}{\sqrt{2}+1} w(OPT(G))$.*

*Proof.* **Notation**: Consider a bipartite subgraph $S \subseteq G$, that satisfies $S = (\mathcal{X}', \mathcal{Y}', E')$, $\mathcal{X}' \subseteq \mathcal{X}$, $\mathcal{Y}' \subseteq \mathcal{Y}$, and $|\mathcal{X}'| = |\mathcal{Y}'|$. Let $Min(S)$ denote a *minimum* weight perfect matching on $S$, and $RSD(S)$ denote the expected weight returned by Algorithm 1 on graph $S$.

For any $x \in \mathcal{X}'$, we use $\lambda(S, x)$ to denote the edge between $x$ and its most preferred agent in $\mathcal{Y}'$. Define $R(S, x)$ as the remaining graph after removing $x$, $x$'s most preferred agent, and all the edges containing $x$ or $x$'s most preferred agent from $S$.

We begin by simply expressing $RSD(S)$ in terms of these quantities.

**Lemma 3.2.2.** *For any subgraph $S$ as decribed above,*
$RSD(S) = \frac{1}{|\mathcal{X}'|} \sum_{x \in \mathcal{X}'} w(\lambda(S, x)) + \frac{1}{|\mathcal{X}'|} \sum_{x \in \mathcal{X}'} RSD(R(S, x))$.

*Proof.* This simply follows from definition of expectation. In the first round of Algorithm 1, an agent $x$ is selected uniformly at random from $\mathcal{X}'$. Given that $x$ is selected, the edge added to the matching is exactly $\lambda(S, x)$, and the expected weight of the matching for the remaining graph is exactly $RSD(R(S, x))$. Each of these occurs with probability $1/|\mathcal{X}'|$. □

We now state the main technical lemma which allows us to prove the result. This lemma gives a bound on the maximum weight matching in terms of the quantities defined above.

**Lemma 3.2.3.** *For any given graph $G = (\mathcal{X}, \mathcal{Y}, E)$, one of the following two cases must be true:*
***Case 1***: $w(OPT(G)) \leq \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} w(OPT(R(x))) + \frac{\sqrt{2}+1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} w(\lambda(x))$
***Case 2***: $w(OPT(G)) \leq (\sqrt{2}+1) w(Min(G))$

We will prove this lemma below, but first we discuss how the rest of the proof will proceed. When Case 1 above holds, we know that at any step of the algorithm, the change in the weight of the optimum solution in the remaining graph is not that different from the weight of the edge selected by our algorithm. This allows us to compare the weight of $OPT$ with the weight of the matching returned by our algorithm. In fact, this is the technique used in a previous paper [19] to analyze RSD for complete graphs (i.e., non-bipartite graphs), and show that RSD gives a 2-approximation for perfect matching on complete graphs. Similar to Case 1 in Lemma 3.2.3, this was done by proving that in each step, the expected loss of the optimal matching is at most twice the expected weight of the chosen edge, and thus the entire algorithm gives a 2-approximation.

It is important to note here that this *does not* work for bipartite graphs. In bipartite matching, using only this method will not give an approximation ratio better than 3. To see this, consider the bipartite graph in Figure 3.1. Suppose $G = (\mathcal{X}, \mathcal{Y}, E)$ is a complete bipartite graph, $|\mathcal{X}| = |\mathcal{Y}| = N$. The edges shown in the Figure are the maximum weight matching of $G$; all the other edges have weight of 1. It is easy to see that these edge weights form a metric. $\forall x \in \mathcal{X}$, $x$'s most preferred agent in $\mathcal{Y}$ is $y_1$, second preferred agent is $y_2$, ..., least preferred agent is $y_n$ (we can always perturb the edge weights by an infinitesimal amount to remove ties for this example). Then the weight of the optimum solution is $w(OPT(G)) = (N-1) + 3$. In this example, the expected decrease in the weight of the optimal matching in the first step of RSD is 3: choosing $x_1$ loses 3, and choosing any other agent $x_i$ in $\mathcal{X}$ loses 3 since $(x_1, y_1)$ and $(x_i, y_i)$ can no longer ne used (decrease of 4), but the edge $(x_1, y_i)$ can be used (increase of 1). On the other hand, the expected weight of the edge chosen by RSD is $\frac{3+(N-1)}{N}$. In this case, almost 3 times the expected weight of the chosen edge is needed to compensate for the loss of optimal matching, so the inequality in Case 1 above only holds if we replace $\sqrt{2}+1$ with 3, and thus would only result in a 3-approximation.

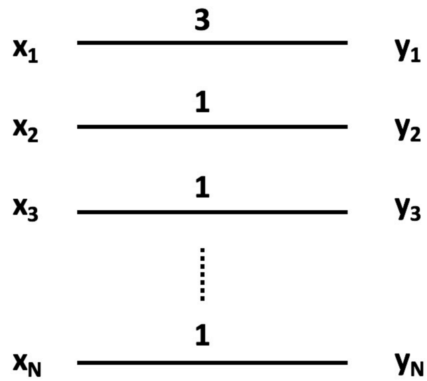


**Figure 3.1: An example graph for RSD.**

We get around this problem by adding Case 2 to our lemma. We are able to show that in any step, either the expected loss of the weight of the optimal matching is at most $(\sqrt{2}+1)$ times of expected weight of the chosen edge, or the weight of the optimal matching is at most $(\sqrt{2}+1)$ times weight of the minimum weight matching. By combining these two cases, we can easily show the following claim which completes the proof of the theorem.

**Proposition 3.2.1.** *As long as Lemma 3.2.3 holds for every $S$, Algorithm 1 provides a $(\sqrt{2}+1)$-approximation to the Maximum weight perfect matching: $RSD(G) \geq \frac{1}{\sqrt{2}+1} w(OPT(G))$.*

*Proof.* We proceed by induction. Clearly when $G$ only has two agents, RSD produces the optimum matching.

Now consider a bipartite graph $G = (\mathcal{X}, \mathcal{Y}, E)$ with $|\mathcal{X}| = |\mathcal{Y}| = N$, and suppose that the claim is true for all smaller graphs, i.e., $\forall x \in \mathcal{X}$, we know that $RSD(R(G,x)) \geq \frac{1}{\sqrt{2}+1} w(OPT(R(G,x)))$.

If Case 2 in Lemma 3.2.3 holds for $G$, then because $Min(G)$ is the minimum weight perfect matching, we know that $w(Min(G)) \leq RSD(G)$. So $RSD(G) \geq \frac{1}{\sqrt{2}+1} w(OPT(G))$. Otherwise Case 1 in Lemma 3.2.3 must be true, i.e.,

$$w(OPT(G)) \leq \frac{1}{N} \sum_{x \in \mathcal{X}} w(OPT(R(G,x))) + \frac{\sqrt{2}+1}{N} \sum_{x \in \mathcal{X}} w(\lambda(G,x))$$

By our assumption,

$$w(OPT(G)) \leq \frac{\sqrt{2}+1}{N} \sum_{x \in \mathcal{X}} RSD(R(G,x)) + \frac{\sqrt{2}+1}{N} \sum_{x \in \mathcal{X}} w(\lambda(G,x))$$

This completes the proof by Lemma 3.2.2. □

We now proceed with the main technical part of the proof, i.e., the proof of Lemma 3.2.3.

**Proof of Lemma 3.2.3** For compactness of notation, since $S$ is fixed, we will omit $S$ and simply write $\lambda(x)$ and $R(x)$ instead of $\lambda(S,x)$ and $R(S,x)$. For any fixed $x \in \mathcal{X}'$, denote $x$'s most preferred agent in $\mathcal{Y}'$ as $y$ (so $\lambda(x) = (x,y)$). In $OPT(S)$, suppose $x$ is matched to $b \in \mathcal{Y}'$, and $y$ is matched to $a \in \mathcal{X}'$. In $Min(S)$, suppose $b$ is matched to $m \in \mathcal{X}'$. $\forall x \in \mathcal{X}'$, there exist $y$, $a$, $b$, $m$ as described above. As shown in Figure 3.2, denote edge $(x,y)$ by $\lambda(x)$, $(x,b)$ by $P(x)$, $(a,y)$ by $\bar{P}(x)$, and $(a,b)$ by $D(x)$.

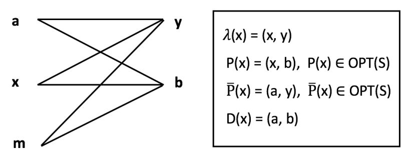


**Figure 3.2: Notation of $\lambda(x)$, $P(x)$, $\bar{P}(x)$, $D(x)$.**

We'll prove Lemma 3.2.3 by showing that if **Case 2** is not true, then **Case 1** must be true. Suppose **Case 2** is not true, i.e., $w(OPT(S)) > (\sqrt{2}+1)w(Min(S))$.

Suppose that random serial dictatorship picks $x \in \mathcal{X}'$. Then $OPT(R(S,x))$ is at least as good as the matching obtained by removing $P(x)$ and $\bar{P}(x)$, and adding $D(x)$ to $OPT(S)$ (the rest stay the same):

$$w(OPT(R(x))) \geq w(OPT(S)) - w(P(x)) - w(\bar{P}(x)) + w(D(x))$$

Note that when $\lambda(x) \in OPT(S)$, $\bar{P}(x) = P(x) = D(x)$, and the inequality still holds. Summing this up over all nodes $x$, we obtain:

$$\frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'} w(OPT(R(x))) \tag{3.1}$$
$$\geq \frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'}(w(OPT(S)) - w(P(x)) - w(\bar{P}(x)) + w(D(x)))$$
$$= w(OPT(S)) - \frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'}(w(P(x)) + w(\bar{P}(x)) - w(D(x)))$$
$$= (1 - \frac{1}{|\mathcal{X}'|})w(OPT(S)) - \frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'}(w(\bar{P}(x)) - w(D(x))) \tag{3.2}$$

In Figure 3.2, by the triangle inequality, we know that

$$w(a,y) \leq w(a,b) + w(m,b) + w(m,y)$$

Because $\lambda(m)$ is the edge to $m$'s most preferred agent, $w(m,y) \leq w(\lambda(m))$, and thus

$$w(\bar{P}(x)) \leq w(D(x)) + w(m,b) + w(\lambda(m)))$$

Summing this up for all $x \in \mathcal{X}'$, note that each $x$ is matched to a unique $b$ in $OPT(S)$, and each $b$ is matched to a unique $m$ in $Min(S)$, so each agent in $\mathcal{Y}'$ appears as $b$ exactly once and each agent in $\mathcal{X}'$ appears as $m$ exactly once.

$$\sum_{x\in\mathcal{X}'} w(\bar{P}(x)) \leq \sum_{x\in\mathcal{X}'} w(D(x)) + w(Min(S)) + \sum_{x\in\mathcal{X}'} w(\lambda(x)))$$
$$\sum_{x\in\mathcal{X}'}(w(\bar{P}(x)) - w(D(x))) \leq w(Min(S)) + \sum_{x\in\mathcal{X}'} w(\lambda(x))) \tag{3.3}$$

Combining Inequality 3.2 and Inequality 3.3,

$$\frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'} w(OPT(R(x)))$$
$$\geq (1 - \frac{1}{|\mathcal{X}'|})w(OPT(S)) - \frac{1}{|\mathcal{X}'|}[w(Min(S)) + \sum_{x\in\mathcal{X}'} w(\lambda(x))] \tag{3.4}$$

$\forall x \in \mathcal{X}'$, $w(P(x)) \leq w(\lambda(x))$ since $\lambda(x)$ is the most preferred edge of $x$, so it is obvious that $w(OPT(S)) \leq \sum_{x\in\mathcal{X}'} w(\lambda(x))$.

By our assumption,

$$w(Min(S)) < \frac{1}{\sqrt{2}+1} w(OPT(S)) \leq \frac{1}{\sqrt{2}+1} \sum_{x \in \mathcal{X}'} w(\lambda(x))$$

Thus, putting this together with Inequality 3.4, we obtain that,

$$\begin{aligned}\frac{1}{|\mathcal{X}'|} \sum_{x \in \mathcal{X}'} w(OPT(R(x))) &\geq w(OPT(S)) - \frac{1}{|\mathcal{X}'|}(2 + \frac{1}{\sqrt{2}+1}) \sum_{x \in \mathcal{X}'} w(\lambda(x))) \\ &= w(OPT(S)) - \frac{\sqrt{2}+1}{|\mathcal{X}'|} \sum_{x \in \mathcal{X}'} w(\lambda(x))\end{aligned}$$

□

### 3.2.1 Partial One-sided Ordinal Preferences

In this section, we consider the case when we are given even less information than in the previous one, i.e., only partial preferences. We begin by establishing the following easy result for the algorithm that produces a completely random matching.

**Algorithm 2:** Random Algorithm for Perfect Bipartite Matching.

Initialize $M = \emptyset$, $G = (\mathcal{X}, \mathcal{Y}, E)$ ;
**while** $E \neq \emptyset$ **do**
    Pick an edge $e = (x, y)$ from $E$ uniformly at random and add it to $M$ ;
    Remove $x$, $y$, and all edges containing $x$ or $y$ from $G$ ;
**end**
**Final Output:** Return $M$.

**Lemma 3.2.4.** *[Lemma 2.5 in [18]] Suppose $G = (\mathcal{X}, \mathcal{Y}, E)$ is a complete bipartite graph on the set of nodes $\mathcal{X}, \mathcal{Y}$ with $|\mathcal{X}| = |\mathcal{Y}| = N$. Then, the expected weight of the random perfect matching returned by Algorithm 2 for the input $G$ is $\mathbb{E}[w(M)] = \frac{1}{N} \sum_{(x,y) \in E} w(x, y)$.*

This lemma was proved in [18]. The expected weight of the random matching is calculated by considering the probability of a given edge in the random matching (every edge is present with equality probability).

**Theorem 3.2.5.** *The uniformly random perfect matching is a 3-approximation to the maximum-weight matching.*

*Proof.* Let $OPT$ be the optimal perfect matching. Suppose $(x, y)$ is an edge in $OPT$. Then for any edge $(a, b) \in E$ , by the triangle inequality,

$$w(x, y) \leq w(x, b) + w(a, y) + w(a, b)$$

Summing up for all $(a, b) \in E$ ,

$$N^2 w(x, y) \leq N \sum_{b \in \mathcal{Y}} w(x, b) + N \sum_{a \in \mathcal{X}} w(a, y) + \sum_{(a,b) \in E} w(a, b)$$

Summing up for all $(x, y) \in OPT$,

$$\begin{aligned} N^2 w(OPT) &\leq N \sum_{(a,b) \in E} w(a, b) + N \sum_{(a,b) \in E} w(a, b) + N \sum_{(a,b) \in E} w((a, b) \\ &= 3N \sum_{(a,b) \in E} w(a, b) \end{aligned}$$

Let $M$ be the matching returned by Algorithm 2. Then, by Lemma 3.2.4,

$$\mathbb{E}[w(M)] = \frac{1}{N} \sum_{(a,b) \in E} w(a, b) \geq \frac{1}{3} w(OPT)$$

□

The following algorithm combines RSD and the random algorithm for the case that we are only given the top $\alpha N$ ordinal preferences for agents in $\mathcal{X}$ in the one-sided model.

---

**Algorithm 3:** Algorithm for Perfect Matching given partial one-sided ordering.

---

Run Algorithm 1, stop when $|M| = \alpha N$, then form random matches until all agents are matched. Return $M$.

---

**Theorem 3.2.6.** *Suppose there is a strict preference ordering $P_x$ over the agents in $\mathcal{Y}$ for each agent $x \in \mathcal{X}$. We are only given top $\alpha N$ agents in $P_x$ in order. Then, the expected weight of the perfect matching $M$ returned by Algorithm 3 is $\mathbb{E}[w(M)] \geq \frac{1}{3-(2-\sqrt{2})\alpha} w(OPT(G))$, as shown in Figure 1.1.*

*Proof.* We use the same notation as in the proof of Theorem 3.2.1. We apply our main technical result (Lemma 3.2.3) to analyze this algorithm. Define $Alg_i(S)$ to be the expected weight of the chosen edge in round $i$ of RSD on any subgraph $S$. For any bipartite graph $S$, let $Rand(S)$ denote the expected weight of the perfect matching returned by Algorithm 2, and $Avg(S)$ denote the average weight of edges in $S$.

We begin by bounding $w(OPT(G))$ by the sum of the expected weights of the chosen edges in RSD, and the weight of the remaining subgraph.

**Lemma 3.2.7.** *Let $L(G,\ell)$ be the subgraph of $G$ after $\ell$ rounds of RSD, which has $N-\ell$ nodes both in $\mathcal{X}$ and $\mathcal{Y}$. Note that $L(G,\ell)$ is a random variable. Then we have that:*

$$w(OPT(G)) \leq (\sqrt{2}+1)\sum_{i=1}^{\ell} Alg_i(G) + 3\mathbb{E}[Rand(L(G,\ell))]$$

*Proof.* We prove this by induction on $\ell$. For the Base Case, when $\ell = 0$, then this simply reduces to Theorem 3.2.5. Now assume by the inductive hypothesis that, $\forall x \in \mathcal{X}$,

$$w(OPT(R(G,x))) \leq (\sqrt{2}+1)\sum_{i=1}^{\ell-1} Alg_i(R(G,x)) + 3\mathbb{E}[Rand(L(R(G,x),\ell-1))]$$

If Case 1 in Lemma 3.2.3 holds for $G$, then

$$\begin{aligned} w(OPT(G)) &\leq \frac{\sqrt{2}+1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} w(\lambda(G,x)) + \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} w(OPT(R(G,x))) \\ &\leq (\sqrt{2}+1)\sum_{i=1}^{\ell} Alg_i(G) + 3\mathbb{E}[Rand(L(G,\ell))] \end{aligned}$$

The last inequality is simply because of the inductive hypothesis, and the fact that $\mathbb{E}[Rand(L(G,\ell))] = \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \mathbb{E}[Rand(L(R(G,x),\ell-1))]$. If instead Case 2 in Lemma 3.2.3 holds for $G$, then

$$w(OPT(G)) \leq (\sqrt{2}+1)w(Min(G))$$

Let's consider a perfect matching on $G$ generated by running RSD for $\ell$ rounds, and then obtaining the minimum weight matching for the remaining subgraph. By the definition of

$Min(G)$, the weight of the matching described above is no less than $Min(G)$:

$$\begin{aligned} w(OPT(G)) &\leq (\sqrt{2}+1)w(Min(G)) \\ &\leq (\sqrt{2}+1)\sum_{i=1}^{\ell} Alg_i(G) + (\sqrt{2}+1)\mathbb{E}[w(Min(L(G,\ell)))] \\ &\leq (\sqrt{2}+1)\sum_{i=1}^{\ell} Alg_i(G) + (\sqrt{2}+1)\mathbb{E}[Rand(L(G,\ell))] \\ &\leq (\sqrt{2}+1)\sum_{i=1}^{\ell} Alg_i(G) + 3\mathbb{E}[Rand(L(G,\ell))] \end{aligned}$$

□

To finish the proof of the theorem, we need to be able to compare $Rand(L(G,\ell))$ and $Alg_i$. After all, if the random part of our matching is much larger in weight than the RSD part, then the random part will dominate, resulting in only a 3 approximation. Fortunately, it is not hard to see the following lemma. Let $G' = L(G, \alpha N)$ be a random variable representing the graph obtained by running RSD on $G$ for $\alpha N$ rounds, which we can always do if we are given the top $\alpha N$ preferences of every agent.

**Lemma 3.2.8.** *$\forall i \leq \alpha N$, the weight of $Alg_i(G)$ is greater than or equal to the expected average edge weight in $G'$, i.e., $Alg_i(G) \geq \mathbb{E}[Avg(G')]$.*

*Proof.* First notice that $Alg_1(G) \geq Alg_2(G)$. This is true because:

$$\begin{aligned} Alg_2(G) &= \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|-1}\sum_{y\in\mathcal{X}-x} w(\lambda(R(G,x),y)) \\ &\leq \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|-1}\sum_{y\in\mathcal{X}-x} w(\lambda(G,y)) \\ &= \frac{|\mathcal{X}|-1}{|\mathcal{X}|(|\mathcal{X}|-1)}\sum_{y\in\mathcal{X}} w(\lambda(G,y)) \\ &= Alg_1(G) \end{aligned}$$

The inequality above is simply because the best edge leaving $y$ in a smaller graph $R(G,x)$ is at most the best edge leaving it in a larger graph $G$. By the same argument, we know that $Alg_i(G) \geq Alg_{i+1}(G)$ for all $i$.

Now consider an arbitrary complete graph $S = (\mathcal{X}', \mathcal{Y}', E')$ with $|\mathcal{X}'| = |\mathcal{Y}'|$. One way to think of $Avg(S)$ is as an expected value of the following randomized algorithm: take a node $x$ in $\mathcal{X}'$ uniformly at random, and then take a random edge leaving that node, and return its weight. The expected value returned by this algorithm is exactly the expected weight of an edge in $S$ taken uniformly at random, i.e., exactly $Avg(S)$. Compare this algorithm with the performance of RSD; RSD does exactly the same thing in the first round, but chooses the best edge coming out of $x$ instead of a random edge. Therefore, the first round of RSD on any graph always performs better than the average edge weight. In particular, this is true for every instantiation of the graph $G'$, and thus $Alg_{\alpha N+1}(G) \geq \mathbb{E}[Avg(G')]$. This concludes the proof.

□

Finally, let's finish the proof of Theorem 3.2.6. By Lemma 3.2.7,

$$\begin{aligned} w(OPT(G)) &\leq (\sqrt{2}+1)\sum_{i=1}^{\alpha N} Alg_i(G) + 3\mathbb{E}[Rand(G')] \\ &= (\sqrt{2}+1)\sum_{i=1}^{\alpha N} Alg_i(G) + 3(1-\alpha)N \times \mathbb{E}[Avg(G')], \end{aligned}$$

By Lemma 3.2.8,

$$\sum_{i=1}^{\alpha N} Alg_i(G) \geq \alpha N \times \mathbb{E}[Avg(G')],$$

and thus,

$$\begin{aligned} w(OPT(G)) &\leq (3-(2-\sqrt{2})\alpha)(\sum_{i=1}^{\alpha N} Alg_i(G) + (1-\alpha)N \times \mathbb{E}[Avg(G')]) \\ &= (3-(2-\sqrt{2})\alpha)(\sum_{i=1}^{\alpha N} Alg_i(G) + \mathbb{E}[Rand(G')]) \end{aligned}$$

Note that $\sum_{i=1}^{\alpha N} Alg_i(G) + \mathbb{E}[Rand(G')]$ is the expected weight of $M$, which completes the proof:

$$w(OPT(G)) \leq (3-(2-\sqrt{2})\alpha)\mathbb{E}[w(M)].$$

□

## 3.3 Two-sided Ordinal Preferences

For two-sided preferences, we give separate algorithms for the cases when $\alpha \geq \frac{1}{2}$ and when $\alpha \leq \frac{1}{2}$, as these require somewhat different techniques.

$\boldsymbol{\alpha \geq \frac{1}{2}}$ While for the case when $\alpha < \frac{1}{2}$ new techniques are necessary to obtain a good approximation, the approach for the case when $\alpha \geq \frac{1}{2}$ is essentially the same as the one used in [18]. We adopt this approach to deal with bipartite graphs and with partial preferences, giving us a 1.8-approximation for $\alpha = 1$. To do this, we re-state the definition of Undominated Edges from [18], and a standard greedy algorithm for forming a matching of size $k$.

**Definition 3.3.1.** *(Undominated Edges) Given a set $E$ of edges, $(x, y) \in E$ is said to be an undominated edge if for all $(x, a)$ and $(y, b)$ in $E$, $w(x, y) \geq w(x, a)$ and $w(x, y) \geq w(y, b)$.*

Note that an undominated edge must always exist: either there are two nodes $x$ and $y$ such that they are each other's top preferences (and so $(x, y)$ is undominated), or there is a cycle $x_1, x_2, \ldots$ in which $x_{i+1}$ is the top preference of $x_i$, in which case all edges in the cycle must be the same weight, and thus all edges in the cycle are undominated. This also gives us an algorithm for determining if an edge $(x, y)$ is undominated: either $x$ and $y$ prefer each other over all other agents, or it is part of such a cycle of top preferences.

**Lemma 3.3.1.** *Given an edge set $E$ of a complete bipartite graph $G = (\mathcal{X}, \mathcal{Y}, E)$, the weight of any undominated edge is at least one third as much as the weight of any other edge in $E$, i.e., if $e = (x, y)$ is an undominated edge in $E$, that $x \in \mathcal{X}$, $y \in \mathcal{Y}$, then for any $(a, b) \in E$, $a \in \mathcal{X}$, $b \in \mathcal{Y}$, $w(x, y) \geq \frac{1}{3} w(a, b)$.*

*Proof.* Since $e = (x, y)$ is an undominated edge, $w(x, y) \geq w(x, b)$ and $w(x, y) \geq w(a, y)$. By the triangle inequality, we know that $w(a, b) \leq w(x, y) + w(x, b) + w(a, y) \leq 3w(x, y)$. □

Before stating the full algorithm for the case when $\alpha \geq \frac{1}{2}$, we mention two lemmas which will be useful to establish its approximation ratio. These lemmas are essentially the same as the similar ones from [18], except that we must adjust all the factors to deal with bipartite

**Algorithm 4:** Greedy Algorithm for Max $k$-Matching of two-sided ordering.

Initialize $M = \emptyset$, $E$ is the valid set of edges initialized to the complete bipartite graph $G$ ;
**while** $E \neq \emptyset$ **do**
  Pick an undominated edge $e = (x, y)$ from E and add it to $M$ ;
  Remove $x$, $y$, and all edges containing $x$ or $y$ from $E$ ;
  **if** $|M| = k$ **then**
    break ;
  **end**
**end**
**Final Output:** Return $M$.

graphs, while [18] considered only non-bipartite graphs. The basic analysis techniques remain the same, however, and we only provide proofs of these lemmas for completeness.

**Lemma 3.3.2.** *[Adjusted from Lemma 2.2 in [18]] Suppose $G = (\mathcal{X}, \mathcal{Y}, E)$ is a complete bipartite graph on the set of nodes $\mathcal{X}, \mathcal{Y}$ with $|\mathcal{X}| = |\mathcal{Y}| = N$. Given $k = \gamma N$, the performance of the greedy $k$-matching returned by Algorithm 4 with respect to the optimal perfect matching OPT is given by $\frac{3-2\gamma}{\gamma}$.*

*Proof.* The analysis here is essentially identical to that of a similar lemma in [18], except that Lemma 3.3.1 gives a ratio of 3 instead of 2 between any edge and an undominated edge for bipartite graphs. We include the whole analysis of the framework for completeness.

Let $M$ be the greedy $k$-matching, and $M^*$ be the optimal perfect matching. We show the claim by charging every edge in $M^*$ to one or more edges in the greedy matching $M$. Consider any edge $e^* = (a, b)$ in $M^*$, the edge must belong to one of the following two types.

1. (Type I) Some edges consisting of $a$ or $b$ (both $a$ and $b$) are present in $M$.

2. (Type II) No edge in $M$ has $a$ or $b$ as an endpoint.

Suppose that $M^*$ contains $m_1$ Type I edges, and $m_2$ Type II edges. We know that $m_1 + m_2 = N$. Let $T \subset M$ denote the heaviest $\frac{m_1}{2}$ edges in $M$. Initialize $U$ as all the edges in $M$. We describe our charging algorithm in three phases.

(First Phase) We can charge all Type I edges in $M^*$ to the edges in $T$, so that $\sum_{e \in T} s_e w_e \geq \sum_{e \in TypeI(M^*)} w_e$, $s_e \leq 2$. We charge the edges as follows: Repeat until $U$ contains no Type I edge: pick a type I edge $e^* = (a, b)$ from $U$. Suppose that $e = (a, c)$ is the first edge containing either a or b that was added to $M$, Since $w_e \geq w^*_e$, charge $e^*$ to $e$, increase $s_e$ by one and remove $e^*$ from U. In the end, all the edges that are charged in $M$ have $s_e \leq 2$, and $\sum_e s_e = m_1$ . We can transfer the slots to the heaviest $\frac{m_1}{2}$ edges in $M$, each has $s_e \leq 2$. Keep transfering the slots to the heaviest $\frac{m_1}{\mu}$ edges in $M$, so that each edge has $s_e \leq \mu$.

(Second Phase) Repeat until $s_e = \mu$ for all $e \in M \backslash T$ or until $U$ is empty: pick any arbitrary edge $e^*$ from $U$ and the smallest edge $e \in M \backslash T$ such that $s_e < \mu$ By Lemma 3.3.1 $w_e* \leq 3w_e$, charge $e*$ using three slots of $e$, transfer slots to the heaviest edges $e \in M \backslash T$ such that $s_e < \mu$. So $e^*$ is charged by three slot from edges in $M \backslash T$.

At the end of the second phase, $|U| = \max(0, m_2 - (k - \frac{m_1}{\mu}) \times \frac{\mu}{3})$.

(Third Phase) Repeat until $U$ is empty: pick any arbitrary edge $e^*$ from $U$. Since $w_e^* \leq 3w_e$ for all $e \in M$, charge $e^*$ uniformly to all edges in $M$, i.e., increase $s_e$ by $\frac{3}{k}$ for every $e \in M$ and remove $e^*$ from $U$.

At the end of the third phase, for every $e \in M$,

$$s_e \leq \mu + \frac{3}{k} \max(0, m_2 - (k - \frac{m_1}{\mu}) \times \frac{\mu}{3})$$

Because $m_1 + m_2 = N$,

$$s_e \leq \max(\mu, \frac{2m_2 + N}{k})$$

Type II edges don't share nodes with any of the $k$ edges in $M$, so $m_2 + k \leq N$,

$$s_e \leq \max(\mu, \frac{3N - 2k}{k})$$

$$s_e \leq max(\mu, \frac{3 - 2\gamma}{\gamma})$$

Let $\mu = \frac{3-2\gamma}{\gamma}$,

$$s_e \leq \frac{3 - 2\gamma}{\gamma}$$

□

**Lemma 3.3.3.** *[Adjusted from Lemma 2.6 in [18]] Let $G_T = (\mathcal{X}_T, \mathcal{Y}_T, E_T)$ be a complete bipartite subgraph on the set of nodes $\mathcal{X}_T \subseteq \mathcal{X}$, $\mathcal{Y}_T \subseteq \mathcal{Y}$, with $|\mathcal{X}_T| = |\mathcal{Y}_T| = n$, and let $M$ be any perfect matching on $G = (\mathcal{X}, \mathcal{Y}, E)$. Then, the following is an upper bound on the* *weight of $M$,*

$$nw(M) \leq (2 + \frac{N}{n}) \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x, y) + \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y} \setminus \mathcal{Y}_T}} w(x, y) + \sum_{\substack{x \in \mathcal{X} \setminus \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x, y)$$

*Proof.* For $e = (x, y) \in M, e' = (a, b) \in E_T$, by the triangle inequality,

$$w(a, y) + w(a, b) + w(x, b) \geq w(x, y)$$

Sum up for all $(a, b) \in E_T$,

$$n \sum_{a \in \mathcal{X}_T} w(a, y) + \sum_{\substack{a \in \mathcal{X}_T \\ b \in \mathcal{Y}_T}} w(a, b) + n \sum_{b \in \mathcal{Y}_T} w(x, b) \geq n^2 w(x, y)$$

Sum up for all $(x, y) \in M$,

$$n \sum_{\substack{a \in \mathcal{X}_T \\ y \in \mathcal{Y}}} w(a, y) + N \sum_{\substack{a \in \mathcal{X}_T \\ b \in \mathcal{Y}_T}} w(a, b) + n \sum_{\substack{b \in \mathcal{Y}_T \\ x \in \mathcal{X}}} w(x, b) \geq n^2 w(M)$$

Because $\mathcal{Y} = \mathcal{Y}_T \cup \{\mathcal{Y} \setminus \mathcal{Y}_T\}$, and $\mathcal{X} = \mathcal{X}_T \cup \{\mathcal{X} \setminus \mathcal{X}_T\}$,

$$\begin{aligned} & n(\sum_{\substack{a \in \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(a, y) + \sum_{\substack{a \in \mathcal{X}_T \\ y \in \mathcal{Y} \setminus \mathcal{Y}_T}} w(a, y)) + N \sum_{\substack{a \in \mathcal{X}_T \\ b \in \mathcal{Y}_T}} w(a, b) \\ & + n(\sum_{\substack{b \in \mathcal{Y}_T \\ x \in \mathcal{X}_T}} w(x, b) + \sum_{\substack{b \in \mathcal{Y}_T \\ x \in \mathcal{X} \setminus \mathcal{X}_T}} w(x, b)) \\ & \geq n^2 w(M) \end{aligned}$$

Replace $a$ with $x$, and $b$ with $y$,

$$\begin{aligned} & 2n \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x, y) + n \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y} \setminus \mathcal{Y}_T}} w(x, y) + n \sum_{\substack{x \in \mathcal{X} \setminus \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x, y) + N \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x, y) \\ & \geq n^2 w(M) \end{aligned}$$

$$nw(M) \leq (2 + \frac{N}{n}) \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x,y) + \sum_{\substack{x \in \mathcal{X}_T \\ y \in \mathcal{Y} \setminus \mathcal{Y}_T}} w(x,y) + \sum_{\substack{x \in \mathcal{X} \setminus \mathcal{X}_T \\ y \in \mathcal{Y}_T}} w(x,y)$$

□

We can now state the algorithm for $\alpha \geq \frac{1}{2}$. The algorithm is a mix of greedy and random algorithms: for graph $G = (\mathcal{X}, \mathcal{Y}, E)$, given top $\alpha N$ of $P(\mathcal{X})$ and top $\alpha N$ of $P(\mathcal{Y})$, run Algorithm 4 on $k = \alpha N$, to obtain the matching $M_0$. This is possible using the preference we are given. One method we could use at this point is to form a random matching on the rest of the agents. However, this will not give a good approximation, as there are examples when all the high-weight edges are between nodes matched in $M_0$ and nodes which are unmatched. Another method is to randomly choose some matched nodes from $M_0$, make them unmatched, and form a random bipartite matching between all the agents which were not matched in $M_0$, and the nodes which we chose from $M_0$ to become unmatched. This second method is likely to add high-weight edges between nodes in $M_0$ and nodes outside of it to our matching. Mixing over these two methods actually returns a high-weight matching in expectation.

Note that for $\alpha > \frac{3}{4}$ this algorithm does not seem to provide better guarantees than for $\alpha = \frac{3}{4}$. Because of this, for $\alpha > \frac{3}{4}$, we simply run the same algorithm for $\alpha = \frac{3}{4}$

**Theorem 3.3.4.** *Algorithm 5 returns a $\frac{(3-2\alpha)(3-\alpha)}{2\alpha^2-3\alpha+3}$-approximation to the maximum-weight perfect matching given two-sided ordering when $\frac{1}{2} \leq \alpha \leq \frac{3}{4}$.*

*Proof.* $|\mathcal{X}_T| = |\mathcal{Y}_T| = \alpha N$, $|\mathcal{X}_B| = |\mathcal{Y}_B| = (1-\alpha)N$.
By Lemma 3.3.2, $w(M_0) \geq \frac{\alpha}{3-2\alpha} OPT$. By Lemma 3.2.4, the perfect matching output by Algorithm 2 on $B$ has expected weight at least $\frac{1}{(1-\alpha)N} w(B)$. Therefore,

$$\mathbb{E}[w(M_1)] \geq \frac{\alpha}{3-2\alpha} OPT + \frac{1}{(1-\alpha)N} w(B)$$

Because $|X_A| = |Y_A| = (1-\alpha)N$, and they are leftover nodes after $(2\alpha - 1)N$ nodes are chosen uniformly at random from $M_0$,

$$\mathbb{E}[w(E_{AB}) + w(E'_{AB})] = \frac{1-\alpha}{\alpha} w(T, B).$$

Recall that $w(T,B)$ is the total weight of all edges between $T$ and $B$. Let $M_{AB}$ be a

**Algorithm 5:** Algorithm for two-sided matching with partial ordinal information ($\frac{1}{2} \leq \alpha \leq \frac{3}{4}$).

**Input :** $\mathcal{X}, \mathcal{Y}$, top $\alpha N$ of $P(\mathcal{X})$, top $\alpha N$ of $P(\mathcal{Y})$
**Output:** Perfect Bipartite Matching M
Initialize $E$ to be complete bipartite graph on $\mathcal{X}, \mathcal{Y}$, and $M_1 = M_2 = \emptyset$ ;
Let $M_0$ be the output returned by Algorithm 4 for $E$, $k = \alpha N$ ;
Let $\mathcal{X}_T$ be the set of nodes in $\mathcal{X}$ matched in $M_0$, $\mathcal{Y}_T$ be the set of nodes in $\mathcal{Y}$ matched in $M_0$, and $T$ be the complete bipartite graph on $\mathcal{X}_T, \mathcal{Y}_T$ ;
Let $\mathcal{X}_B = \mathcal{X} \backslash \mathcal{X}_T$, $\mathcal{Y}_B = \mathcal{Y} \backslash \mathcal{Y}_T$, and $B$ be the complete bipartite graph on $\mathcal{X}_B, \mathcal{Y}_B$;
**First Algorithm**;
$M_1 = M_0 \cup$ (The perfect matching output by Algorithm 2 on $B$);
**Second Algorithm**;
Choose $(2\alpha - 1)N$ edges from $M_0$ uniformly at random and add them to $M_2$ ;
Let $X_A$ be the set of nodes in $\mathcal{X}_T$ and not in $M_2$, $Y_A$ be the set of nodes in $\mathcal{Y}_T$ and not in $M_2$;
Let $E_{AB}$ be the edges of the complete bipartite graph $(X_A, \mathcal{Y}_B)$ and $E'_{AB}$ be the edges of the complete bipartite graph $(\mathcal{X}_B, Y_A)$ ;
Run random bipartite matching on the set of edges in $E_{AB}$ and $E'_{AB}$ separately to obtain perfect bipartite matchings and add the edges returned by the algorithm to $M_2$;
**Final Output:** Return $M_1$ with probability $\frac{3-2\alpha}{3-\alpha}$ and $M_2$ with probability $\frac{\alpha}{3-\alpha}$.

random bipartite matching formed on edges $E_{AB}$ and $E'_{AB}$. By Lemma 3.2.4,

$$\begin{aligned}\mathbb{E}[w(M_{AB})] &= \frac{1}{(1-\alpha)N}\mathbb{E}[w(E_{AB})] + \frac{1}{(1-\alpha)N}\mathbb{E}[w(E'_{AB})] \\ &= \frac{1}{(1-\alpha)N}\mathbb{E}[w(E_{AB}) + w(E'_{AB})] \\ &= \frac{1}{\alpha N} w(T, B)\end{aligned}$$

By Lemma 3.3.3, with $M = OPT, T = B, n = (1-\alpha)N$:

$$(1-\alpha)Nw(OPT) \leq (2 + \frac{1}{1-\alpha})w(B) + w(T, B)$$

$$\begin{aligned}\mathbb{E}[w(M_{AB})] &= \frac{1}{\alpha N} w(T, B) \\ &\geq \frac{1}{\alpha N}((1-\alpha)Nw(OPT) - \frac{3-2\alpha}{1-\alpha}w(B))\end{aligned}$$

$M_2$ contains $\frac{2\alpha-1}{\alpha}$ fraction of edges randomly chosen from $M_0$, together with $M_{AB}$:

$$\begin{aligned}\mathbb{E}[w(M_2)] &= \frac{2\alpha-1}{\alpha} \times \frac{\alpha}{3-2\alpha} w(OPT) + \mathbb{E}[w(M_{AB})] \\ &\geq \frac{2\alpha-1}{3-2\alpha} w(OPT) + \frac{1}{\alpha N}((1-\alpha)Nw(OPT) - \frac{3-2\alpha}{1-\alpha} w(B)) \\ &= \frac{4\alpha^2-6\alpha+3}{\alpha(3-2\alpha)} w(OPT) - \frac{3-2\alpha}{\alpha(1-\alpha)N} w(B)\end{aligned}$$

Return $M_1$ with probability $\frac{3-2\alpha}{3-\alpha}$ and $M_2$ with probability $\frac{\alpha}{3-\alpha}$. Then, the expected weight of our final matching is

$$\frac{3-2\alpha}{3-\alpha}\mathbb{E}[w(M_1)] + \frac{\alpha}{3-\alpha}\mathbb{E}[w(M_2)] \geq \frac{2\alpha^2-3\alpha+3}{(3-2\alpha)(3-\alpha)} w(OPT).$$

□

$\boldsymbol{\alpha \leq \frac{1}{2}}$ Unlike the case for $\alpha \geq \frac{1}{2}$, this case requires different techniques than in [18]. While the techniques above would still work, they will not give us a bound as good as the one we form below. The idea in this section is to do something similar to our one-sided algorithm for partial preferences: run the greedy algorithm for a while, and then switch to random. Unfortunately, if we simply run the greedy Algorithm 4 and then switch to random, this will not give a good approximation. The reason why this is true is that an undominated edge which is picked by the greedy algorithm may be much worse than the average weight of an edge, and so the approximation factor of the random algorithm will dominate, giving only a 3-approximation. Even taking an undominated edge uniformly at random has this problem. We can fix this, however, by picking each undominated edge with an appropriate probability, as described below. Such an algorithm results in matchings which are guaranteed to be better than either RSD or Random, thus allowing us to prove the result.

This algorithm guarantees that an undominated edge is chosen for any $x$ in any bipartite graph $G$. Now, before we reach an undominated edge, the weights of edges are non-decreasing in the order they are checked. Thus whenever a node $x$ is picked, the algorithm adds an undominated edge $(x_1, y_1)$ to the matching which is guaranteed to have higher weight than all edges leaving $x$. Note that it is not possible to apply this algorithm to one-sided matching because the preferences of agents in $\mathcal{Y}$ are not given, and thus we cannot detect which edges

**Algorithm 6:** Algorithm for two-sided matching with partial ordinal information ($0 \leq \alpha \leq \frac{1}{2}$).

**Input :** $\mathcal{X}, \mathcal{Y}$, top $\alpha N$ of $P(\mathcal{X})$ and $P(\mathcal{Y})$
Initialize $M = \emptyset$, $G = (\mathcal{X}, \mathcal{Y}, E)$ ;
**while** $E \neq \emptyset$ **do**
    Pick an agent $x$ uniformly at random from $\mathcal{X}$ ;
    Let $y$ denote $x$'s most preferred agent in $\mathcal{Y}$ ;
    $x_1 \leftarrow x$, $y_1 \leftarrow y$, $c \leftarrow y_1$;
    **while** $(x_1, y_1)$ *is not an undominated edge* **do**
        **if** $c = y_1$ **then**
            $x_1 \leftarrow y_1$'s most preferred agent in $\mathcal{X}$ ;
            $c \leftarrow x_1$;
        **else**
            $y_1 \leftarrow x_1$'s most preferred agent in $\mathcal{Y}$ ;
            $c \leftarrow y_1$;
        **end**
    **end**
    Take $(x_1, y_1)$ from $E$ and add it to $M$ ;
    Remove $x_1$, $y_1$, and all edges containing $x_1$ or $y_1$ from the graph $G$ ;
    **if** $|M| = \alpha N$ **then**
        break;
    **end**
**end**
Run Algorithm 2 for the remaining graph $G$, add the edges returned by the algorithm to $M$. **Final Output:** Return $M$.

are undominated.

**Theorem 3.3.5.** *Algorithm 6 returns a* $(3-\alpha)$*-approximation to the maximum-weight perfect matching given two-sided ordering when* $0 \leq \alpha \leq \frac{1}{2}$.

*Proof.* We use a similar method and the same notation as in Section 3.2 to prove this theorem. Essentially, because we are always picking undominated edges, we can form a linear interpolation between a factor of 2 and a factor of 3 for random matching, instead of between factors $\sqrt{2}+1$ and 3 as for one-sided preferences. The reason why we are able to form such an interpolation is entirely because of the probabilities with which we choose the undominated edges; if we simply chose arbitrary undominated edges or choose them uniformly at random, then there are examples where the random edge weights will dominate and result in a poor approximation, since undominated edges are only guaranteed to be within a factor of 3 of the average edge weight.

Besides those was used in the proof of Theorem 3.2.1, we introduce some new notation. Suppose that Algorithm 6 picks $x \in \mathcal{X}'$, and end up with an undominated edge $(x_1, y_1)$. Let $\lambda_D(S,x)$ denote the undominated edge picked by the algorithm for $x$ in graph $S$, $\lambda_D(S,x) = (x_1, y_1) = \lambda(S, x_1)$ in this case. And let $R_D(S,x)$ denote the remaining graph after removing $\lambda_D(S,x)$ and the edges connected to both vertexes of $\lambda_D(S,x)$.

We start with a lemma to bound the maximum weight matching.

**Lemma 3.3.6.** *For any given subgraph* $S = (\mathcal{X}', \mathcal{Y}', E')$,
$w(OPT(S)) \leq \frac{1}{|\mathcal{X}'|} \sum_{x\in\mathcal{X}'} w(OPT(R_D(S,x))) + \frac{2}{|\mathcal{X}'|} \sum_{x\in\mathcal{X}'} w(\lambda_D(S,x))$.

*Proof.* Using the same notation as in the proofs of Theorems 3.2.1 and 3.2.6, suppose that Algorithm 6 picks $x \in \mathcal{X}'$, and ends up with an undominated edge $(x_1, y_1)$. Then $OPT(R_D(S,x)) = OPT(R(S,x_1))$ is at least as good as the matching obtained by removing $P(x_1)$ and $\bar{P}(x_1)$, and adding $D(x_1)$ to $OPT(S)$ (the remaining edges stay the same):

$$\begin{aligned} w(OPT(R_D(S,x))) &\geq w(OPT(S)) - w(P(x_1)) - w(\bar{P}(x_1)) + w(D(x_1)) \\ &\geq w(OPT(S)) - w(P(x_1)) - w(\bar{P}(x_1)) \end{aligned}$$

Because $\lambda_D(S,x)$ is an undominated edge, $w(\lambda_D(S,x)) \geq P(x_1)$,
$w(\lambda_D(S,x)) \geq \bar{P}(x_1)$,

$$w(OPT(R_D(S,x))) \geq w(OPT(S)) - 2w(\lambda_D(S,x))$$

Summing up for all $x$ in $\mathcal{X}'$,

$$\sum_{x\in\mathcal{X}'} w(OPT(R_D(S,x))) \geq |\mathcal{X}'| w(OPT(S)) - 2 \sum_{x\in\mathcal{X}'} w(\lambda_D(S,x))$$

$$w(OPT(S)) \leq \frac{1}{|\mathcal{X}'|} \sum_{x\in\mathcal{X}'} w(OPT(R_D(S,x))) + \frac{2}{|\mathcal{X}'|} \sum_{x\in\mathcal{X}'} w(\lambda_D(S,x)).$$

□

Then we bound $w(OPT(G))$ by the sum of expected weights of chosen edges in Algorithm 6, and the weight of the remaining subgraph. We still use $Alg_i(S)$ as the expected weight of chosen edge in round $i$, but note that for any $x$, the chosen edge is $\lambda_D(G,x)$ instead of $\lambda(G, x)$ as in Theorem 3.2.6. By an identical argument as in our Lemma 3.2.7, we have that the following holds:

$$w(OPT(G)) \leq 2\sum_{i=1}^{\ell} Alg_i(G) + 3\mathbb{E}[Rand(L(G, \ell))].$$

We need to prove that a version of Lemma 3.2.8 still holds for Algorithm 6, as the edges are chosen differently from RSD in each step. In other words, we need to be able to compare $Rand(L(G, \ell))$ and $Alg_i$. This is where we need to use the fact that each undominated edge is carefully chosen with a specific probability. Let $G' = L(G, \alpha N)$ be a random variable representing the graph obtained by running our greedy algorithm on $G$ for $\alpha N$ rounds, which we can always do if we are given the top $\alpha N$ preferences of every agent.

**Lemma 3.3.7.** *$\forall i \leq \alpha N$, $Alg_i(G)$ is heavier than the expected average edge weight in $G'$, i.e., $Alg_i(G) \geq \mathbb{E}[Avg(G']$.*

*Proof.* We must show that $Alg_1(G) \geq Alg_2(G)$. To see this,

$$\begin{aligned} Alg_2(G) &= \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|-1} \sum_{y\in\mathcal{X}-\lambda_D(G,x)} w(\lambda_D(R_D(G,x), y)) \\ &\leq \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|-1} \sum_{y\in\mathcal{X}-\lambda_D(G,x)} w(\lambda_D(G,y)) \end{aligned}$$

The inequality above is because the undominated edge found after selecting $y$ and then following the agents' top preferences in a smaller graph $R_D(G, x)$ is at most that in a larger graph $G$.

Fix some $x \in \mathcal{X}$, and let $(x_1, y_1)$ be the edge $\lambda_D(G, x)$ be the edge added to the matching if $x$ is picked by our algorithm, and thus $x_1$ is the node removed from $\mathcal{X}$. Note that for the case when $x \neq x_1$, we still have that $w(\lambda_D(G, x)) = w(\lambda_D(G, x_1))$, since if $x_1$ is picked by our algorithm. Then the undominated edge next to it $(x_1, y_1)$ is immediately returned. Therefore, in the sum above, we can replace $w(\lambda_D(G, x))$ (since $x$ still remains in $\mathcal{X} - \lambda_D(G, x)$) with $w(\lambda_D(G, x_1))$, and thus equivalently make the sum be over $\mathcal{X} - x$ instead of over $\mathcal{X} - \lambda_D(G, x)$.

$$\begin{aligned} Alg_2(G) &=\leq \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}}\frac{1}{|\mathcal{X}|-1}\sum_{y\in\mathcal{X}-x} w(\lambda_D(G,y)) \\ &= \frac{|\mathcal{X}|-1}{|\mathcal{X}|(|\mathcal{X}|-1)}\sum_{y\in\mathcal{X}} w(\lambda_D(G,y)) \\ &= Alg_1(G) \end{aligned}$$

By the same argument, we know that $Alg_i(G) \geq Alg_{i+1}(G)$ for all $i$.

All that is left is to compare $Alg_{\alpha N+1}(G)$ with $\mathbb{E}[Avg(G')]$. We know that the first round of RSD on any graph always performs better than the average edge weight. For every $x$ that is chosen uniformly at random in the first step of Algorithm 6, the weight of final chosen edge $\lambda_D(x)$ is no smaller than $\lambda_(x)$. Therefore, the expected weight of the chosen edge in the first round of Algorithm 6 is no smaller than that of RSD, thus better than the average edge weight, $Alg_{\alpha N+1}(G) \geq \mathbb{E}[Avg(G')]$. This concludes the proof. □

Finally, to finish the proof of Theorem 3.3.5. Similarly to the proof of Theorem 3.2.6, it is easy to show that there is a linear tradeoff from 3 to 2-approximation for $\alpha = 0$ to $\alpha = 1$, which gives $w(OPT(G)) \leq (3-\alpha)\mathbb{E}[w(M)]$, in which $M$ is a random variable representing the matching returned by Algorithm 6. □

## 3.4 Total Ordering of Edge Weights

For the setting in which we are given the top $\alpha N^2$ edges of $G$ in order, we prove that for $\alpha = \frac{3}{4}$, we can obtain an approximation of $\frac{5}{3}$ in expectation. For larger $\alpha$, however, more information does not seem to help, and so we simply use the algorithm for $\alpha = \frac{3}{4}$ for any $\alpha > \frac{3}{4}$.

**Algorithm 7:** Greedy Algorithm for Max $k$-Matching given the total ordering of edge weights.

Initialize $M = \emptyset$, $E$ is the valid set of edges initialized to the complete bipartite graph $G$ ;
**while** $E \neq \emptyset$ **do**
  Pick the heaviest edge $e = (x, y)$ from E and add it to $M$ ;
  Remove $x$, $y$, and all edges containing $x$ or $y$ from $E$ ;
  **if** $|M| = k$ **then**
    break ;
  **end**
**end**
**Final Output:** Return $M$.

**Lemma 3.4.1.** *Suppose $G = (\mathcal{X}, \mathcal{Y}, E)$ is a complete bipartite graph on the set of nodes $\mathcal{X}, \mathcal{Y}$ with $|\mathcal{X}| = |\mathcal{Y}| = N$. Given $k = \gamma N$, the performance of the greedy $k$-matching returned by Algorithm 7 with respect to the optimal perfect matching OPT is at least $\gamma$, for $\gamma \leq \frac{1}{2}$.*

*Proof.* Let $M$ be the matching returned by Algorithm 4 for $k = N$. From Lemma 3.3.2, $w(M) \geq \frac{1}{2} w(OPT)$. In the proof of Lemma 3.3.2, each edge in $M$ is charged at most twice by edges of OPT, and there are $N$ charges in total. Transfer all the charges to the highest weight $\frac{N}{2}$ edges in $M$; this tells us that the highest weight $\frac{N}{2}$ edges of $M$ are at least $\frac{1}{2} w(OPT)$. Further transfer all the charges to the highest weight $\gamma N$ edges in $M$; this results in each such edge being charged to $1/\gamma$ times by edges of OPT. Therefore, the highest weight $\gamma N$ edges of $M$ are at least $\frac{1}{\gamma} w(OPT)$ in total.

Same as Algorithm 4, Algorithm 7 also picks an undominated edge each round; the difference is the edges in the matching are picked in non-decreasing order. So Algorithm 7 returns a $k$-matching with the same weight as the highest $\gamma N$ edges in the perfect matching returned by Algorithm 4 on the same graph, which gives at least $\frac{1}{\gamma} w(OPT)$. □

**Lemma 3.4.2.** *Suppose $G = (\mathcal{X}, \mathcal{Y}, E)$ is a complete bipartite graph on the set of nodes $\mathcal{X}, \mathcal{Y}$ with $|\mathcal{X}| = |\mathcal{Y}| = N$. Given the order of the top $\alpha N^2$ edges in the graph, we are able to run greedy $k$-matching by Algorithm 7 for $k = (1 - \sqrt{1 - \alpha})N$.*

*Proof.* In the first step of Algorithm 7, the heaviest edge is taken, and $2N - 1$ edges are removed, so at most $2N - 1$ edges are lost from the top $\alpha N^2$ edges. After the first $k$ steps of Algorithm 7, the total number of removed edges is:

$$\begin{aligned}&2N-1+2(N-1)-1+...+2(N-(k-1))-1\\&=2(N+N-1+...+N-(k-1))-k\\&=2Nk-k^2\end{aligned}$$

Given the order of top $\alpha N^2$ edges, we are able to run Algorithm 7 for at least $k$ steps until $2Nk-k^2=\alpha N^2$. Solving the equation for $k$, $k=(1-\sqrt{1-\alpha})N$. □

The algorithm for bipartite matching with partial ordinal information is similar to that with partial two-sided ordinal information, except that we only need to consider the case that $k \leq \frac{1}{2}N$, i.e., $1-\sqrt{1-\alpha} \leq \frac{1}{2}$, $\alpha \leq \frac{3}{4}$. In two-sided model, we are given the top $\alpha N$ preferences for both sets of agents, and able to run greedy algorithm for $k=\alpha N$. While in the total ordering model, we could only run greedy algorithm for $k=(1-\sqrt{1-\alpha})N$ given the order of the top $\alpha N^2$ edges. Differently from the two-sided model, $alpha$ does not equal the number of agent pairs we are able to match by greedy algorithm in the total ordering model.

**Algorithm 8:** Algorithm for matching given partial total ordering.

**Input :** $\mathcal{X}, \mathcal{Y}$, order of the top $\alpha N^2$ edges in the graph.

**Output:** Perfect Bipartite Matching M

Initialize $E$ to be complete bipartite graph on $\mathcal{X}, \mathcal{Y}$, and $M_1 = M_2 = \emptyset$ ;

Let $M_0$ be the output returned by Algorithm 7 for $E$, $k = (1 - \sqrt{1-\alpha})N$. Let $\alpha_1 = 1 - \sqrt{1-\alpha}$, then $k = \alpha_1 N$ ;

Let $\mathcal{X}_T$ be the set of nodes in $\mathcal{X}$ matched in $M_0$, $\mathcal{Y}_T$ be the set of nodes in $\mathcal{Y}$ matched in $M_0$, and $T$ be the complete bipartite graph on $\mathcal{X}_T, \mathcal{Y}_T$ ;

Let $\mathcal{X}_B$ be the set of nodes in $\mathcal{X}$ not matched in $M_0$, $\mathcal{Y}_B$ be the set of nodes in $\mathcal{Y}$ not matched in $M_0$, and $B$ is the complete bipartite graph on $\mathcal{X}_B, \mathcal{Y}_B$;

**First Algorithm**;

$M_1 = M_0 \cup$ (The perfect matching output by Algorithm 2 on $B$);

**Second Algorithm**;

Choose $(1 - 2\alpha_1)N$ nodes both from $\mathcal{X}_B$ and $\mathcal{Y}_B$ uniformly at random, get the perfect matching output by Algorithm 2 on these nodes and add the results to $M_2$ ;

Let $X_A$ be the set of nodes in $\mathcal{X}_B$ and not in $M_2$, $Y_A$ be the set of nodes in $\mathcal{Y}_B$ and not in $M_2$;

Let $E_{AT}$ be the edges of the complete bipartite graph $(X_A, \mathcal{Y}_T)$ and $E'_{AT}$ be the edges of the complete bipartite graph $(\mathcal{X}_T, Y_A)$ ;

Run random bipartite matching on the set of edges in $E_{AT}$ and $E'_{AT}$ separately to obtain perfect bipartite matchings and add the edges returned by the algorithm to $M_2$;

**Final Output:** Return $M_1$ with probability $\frac{2}{2+\sqrt{1-\alpha}}$ and $M_2$ with probability $\frac{\sqrt{1-\alpha}}{2+\sqrt{1-\alpha}}$.

**Theorem 3.4.3.** *Algorithm 8 returns a* $\frac{2+\sqrt{1-\alpha}}{2-\sqrt{1-\alpha}}$*-approximation to the maximum-weight matching in expectation for* $\alpha \le \frac{3}{4}$*, as shown in Figure 1.1.*

*Proof.* By Lemma 3.4.2, we are able to run Algorithm 7 for $k = (1-\sqrt{1-\alpha})N$. We analyze the algorithm when $\alpha \le \frac{3}{4}$, $\alpha_1 = 1 - \sqrt{1-\alpha} \le \frac{1}{2}$.

$|\mathcal{X}_T| = |\mathcal{Y}_T| = \alpha_1 N$, $|\mathcal{X}_B| = |\mathcal{Y}_B| = (1-\alpha_1)N$.

By Lemma 3.4.1, $w(M_0) \ge \alpha_1 w(OPT)$. By Lemma 3.2.4, the perfect matching output by Algorithm 2 on $B$ has expected weight $\frac{1}{(1-\alpha_1)N}w(B)$. Thus,

$$\mathbb{E}[w(M_1)] \geq \alpha_1 w(OPT) + \frac{1}{(1-\alpha_1)N}w(B).$$

The analysis of $\mathbb{E}[w(M_2)]$ is very similar to the case when $\alpha_1 \geq \frac{1}{2}$ for Algorithm 5, except that now $B$ is larger than $T$, and so we form a random bipartite matching using all of the nodes in $T$ instead of just some of them. Formally, because $|X_A| = |Y_A| = \alpha_1 N$, and they are leftover nodes after $(1-2\alpha_1)N$ nodes are chosen uniformly at random from $B$, we know that

$$\mathbb{E}[w(E_{AT}) + w(E'_{AT})] = \frac{\alpha_1}{1-\alpha_1}w(T,B).$$

Let $M_{AT}$ be the random bipartite matching formed between sets $A$ and $T$.

By Lemma 3.2.4,

$$\begin{aligned}\mathbb{E}[w(M_{AT})] &= \frac{1}{\alpha_1 N}\mathbb{E}[w(E_{AT})] + \frac{1}{\alpha_1 N}\mathbb{E}[w(E'_{AT})] \\ &= \frac{1}{(1-\alpha_1)N}w(T,B)\end{aligned}$$

By Lemma 3.3.3, setting $M = OPT, T = B, n = (1-\alpha_1)N$,

$$(1-\alpha_1)Nw(OPT) \leq (2 + \frac{1}{1-\alpha_1})w(B) + w(T,B).$$

Thus,

$$\begin{aligned}\mathbb{E}[w(M_{AT})] &= \frac{1}{(1-\alpha_1)N}w(T,B) \\ &\geq \frac{1}{(1-\alpha_1)N}((1-\alpha_1)Nw(OPT) - \frac{3-2\alpha_1}{1-\alpha_1}w(B))\end{aligned}$$

$$\begin{aligned}\mathbb{E}[w(M_2)] &= \frac{1-2\alpha_1}{1-\alpha_1} \times \frac{1}{(1-\alpha_1)N} w(B) + \mathbb{E}[w(M_{AT})] \\ &\geq \frac{1-2\alpha_1}{(1-\alpha_1)^2 N} w(B) + \frac{(1-\alpha_1)Nw(OPT) - \frac{3-2\alpha_1}{1-\alpha_1} w(B)}{(1-\alpha_1)N} \\ &= w(OPT) - \frac{2}{(1-\alpha_1)^2 N} w(B)\end{aligned}$$

Return $M_1$ with probability $\frac{2}{3-\alpha_1} = \frac{2}{2+\sqrt{1-\alpha}}$, and $M_2$ with probability $\frac{1-\alpha_1}{3-\alpha_1} = \frac{\sqrt{1-\alpha}}{2+\sqrt{1-\alpha}}$,

$$\begin{aligned}\frac{2}{3-\alpha_1}\mathbb{E}[w(M_1)] + \frac{1-\alpha_1}{3-\alpha_1}\mathbb{E}[w(M_2)] &\geq \frac{1+\alpha_1}{3-\alpha_1} w(OPT) \\ &= \frac{2-\sqrt{1-\alpha}}{2+\sqrt{1-\alpha}} w(OPT)\end{aligned}$$

□

## 3.5 One-sided Preferences with Restricted Edge Weights

In previous sections, we made the assumption that the agents lie in a metric space, and thus the edge weights, although unknown to us, must follow the triangle inequality. In this section we once again consider the most restrictive type of agent preferences — that of one-sided preferences — but now instead of assuming that agents lie in a metric space, we instead consider settings where edges weights cannot be infinitely different from each other. This applies to settings where the agents are at least somewhat indifferent and the items are somewhat similar; the least-preferred agent and the most-preferred items differ only by a constant factor to any agent. Indeed, when for example purchasing a house in a reasonable market (i.e., once houses that almost no one would buy have been removed from consideration), it is unlikely that any agent would like house $x$ so much more than house $y$ that they would be willing to pay hundreds of times more for $x$ than for $y$.

More formally, for each agent $i \in \mathcal{X}$, we are given a strict preference ordering $P_i$ over the agents in $\mathcal{Y}$. In this section we assume that the highest weight edge $e_{max}$ is at most $\beta$ times of the lowest weight edge $e_{min}$. We normalize the lowest weight edge $e_{min}$ in the graph to $w(e_{min}) = 1$; then for any edge $e \in E$, $w(e) \leq \beta$. We use similar analysis as in Section 3.2, except that instead of getting bounds by using the triangle inequality, the relationships among edge weights are bounded by our assumption of the highest and lowest weight edge ratio. As stated above, we no longer assume the agents lie in a metric space in this section.

**Theorem 3.5.1.** *Suppose* $w(e_{min}) = 1$, $\forall e \in E$, $w(e) \leq \beta$. *The expected weight of the perfect matching returned by Algorithm 1 is* $w(M) \geq \frac{1}{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}} w(OPT)$.

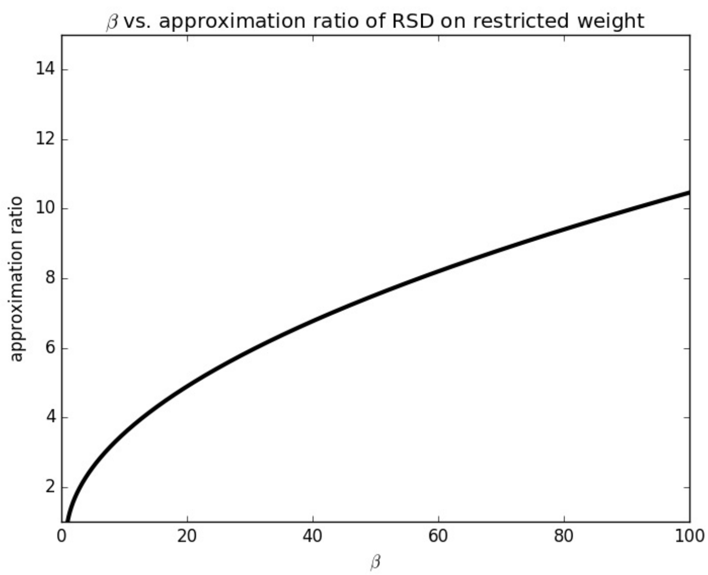


**Figure 3.3: $\beta$ vs. approximation ratio of RSD on restricted weight bipartite graph. For edges with a small difference in weight, we still obtain a reasonable approximation to the optimum matching.**

*Proof.* We use the same notation as in Section 3.2. Once again, our proof relies on the following claim, similar to Lemma 3.2.3. Once the statement below is proven, the rest of the proof proceeds exactly as in Theorem 3.2.1, simply replacing $\sqrt{2} + 1$ with $\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}$.

**Lemma 3.5.2.** *For any given subgraph* $S = (\mathcal{X}', \mathcal{Y}', E')$, *one of the following two cases must be true:*

***Case 1***, $w(OPT(S)) \leq \frac{1}{|\mathcal{X}'|} \sum_{x \in \mathcal{X}'} w(OPT(R(S, x))) + \frac{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}}{|\mathcal{X}'|} \sum_{x \in \mathcal{X}'} w(\lambda(x))$

***Case 2***, $w(OPT(S)) \leq (\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}) w(Min(S))$

*Proof.* Again, we use the same notation as in Section 3.2.

We'll prove Lemma 3.5.2 by showing that if **Case 2** is not true, then **Case 1** must be true. Suppose **Case 2** is not true, $w(OPT(S)) > (\sqrt{\beta - \frac{3}{4}} + \frac{1}{2})w(Min(S))$.

Suppose that random serial dictatorship picks $x \in \mathcal{X}'$. Just as in the proof of Lemma 3.2.3, we obtain that

$$\begin{aligned} &\frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'} w(OPT(R(x))) \\ &\geq (1 - \frac{1}{|\mathcal{X}'|})w(OPT(S)) - \frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'}(w(\bar{P}(x)) - w(D(x))) \end{aligned} \tag{3.5}$$

We know that $\forall e \in E'$, $1 \leq w(e) \leq \beta$. So $w(D(x)) \geq 1$, $w(\bar{P}(x)) \leq \beta$, and thus

$$\frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'}(w(\bar{P}(x)) - w(D(x))) \leq \beta - 1 \tag{3.6}$$

$\forall x \in \mathcal{X}'$, $w(P(x)) \leq w(\lambda(x))$, so it is obvious that $w(OPT(S)) \leq \sum_{x\in\mathcal{X}'} w(\lambda(x))$.

$Min(S)$ is a perfect matching, so $w(Min(S)) \geq |\mathcal{X}'|$. By our assumption,

$$|\mathcal{X}'| \leq w(Min(S)) < \frac{1}{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}} w(OPT(S)) \tag{3.7}$$

Combining Inequalities 3.5, 3.6, and 3.7,

$$\begin{aligned} &\frac{1}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'} w(OPT(R(x))) \\ &\geq w(OPT(S)) - \frac{1}{|\mathcal{X}'|}w(OPT(S)) - \frac{1}{|\mathcal{X}'|}(\beta - 1)|\mathcal{X}'| \\ &\geq w(OPT(S)) - \frac{1}{|\mathcal{X}'|}w(OPT(S)) - \frac{1}{|\mathcal{X}'|}(\beta - 1)\frac{1}{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}}w(OPT(S)) \\ &= w(OPT(S)) - \frac{1}{|\mathcal{X}'|}(1 + \frac{\beta - 1}{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}})w(OPT(S)) \\ &= w(OPT(S)) - \frac{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}}{|\mathcal{X}'|}w(OPT(S)) \\ &\geq w(OPT(S)) - \frac{\sqrt{\beta - \frac{3}{4}} + \frac{1}{2}}{|\mathcal{X}'|}\sum_{x\in\mathcal{X}'} w(\lambda(x)) \end{aligned}$$

□

This completes the proof of the theorem. □

## 3.6 Lower Bound Examples

In this section, we provide some examples to study the lower bounds of algorithms on the maximum weight bipartite matching, given two-sided or one-sided ordinal information.

### 3.6.1 Lower Bound of Two-sided Ordinal Information

**Example** Consider a bipartite graph $G = (\mathcal{X}, \mathcal{Y}, E)$, $\mathcal{X} = \{a, b\}$, $\mathcal{Y} = \{c, d\}$. Let $\epsilon$ be a very small positive number. Consider two sets of weight assignment that have the same two-sided ordinal preferences in metric space: $W1 : w(a, c) = w(b, d) = 1 + \epsilon$, $w(b, c) = 3$, $w(a, d) = 1$. $W2 : w(a, c) = w(b, d) = 1 - \epsilon$, $w(b, c) = 1$, $w(a, d) = \epsilon$. The maximum weight perfect matching for $W1$ is $M1 = \{(a, d), (b, c)\}$, while for $W2$ is $M2 = \{(a, c), (b, d)\}$. Applying any randomized algorithm choosing $M1$ with probability $p$ and $M2$ with probability $1 - p$ to these two weight settings, the optimal algorithm is when $p = \frac{1}{2}$, gives a 1.33-approximation. This is because the expected weight of the matching is $\frac{1}{2}(w(OPT) + \frac{1}{2}w(OPT)) = \frac{3}{4}w(OPT)$, so the approximation ratio is $\frac{w(OPT)}{\frac{3}{4}w(OPT)} = 1.33$.

### 3.6.2 Lower Bound of One-sided Ordinal Information

**Example** For one-sided ordinal information, consider a graph $G = (\mathcal{X}, \mathcal{Y}, E)$, $|\mathcal{X}| = |\mathcal{Y}| = N$, $\mathcal{X} = \{x_1, x_2, ...x_N\}$, $\mathcal{Y} = \{y_1, y_2, ..., y_N\}$. Each agent in $\mathcal{X}$ have the same preferences over agents in $\mathcal{Y}$ as $y_1 > y_2 > ... > y_N$, because of this setting, no algorithm could distinguish agents and get a better performance than random algorithm. Assign the weights of the graph as: for a certain number $\nu \in [0, 1]$, when $i \leq \nu N$, $w(x_i, y_j) = 3$ for $j \leq i$, all other edges have weight 1. The maximum matching is $\{(a_1, b_1), (a_2, b_2), ..., (a_N, b_N)\}$, with a total weight $(2\nu + 1)N$. Random matching of this graph gets an expected weight of $(\nu(\frac{1}{N} + \nu) + 1)N$, when $N$ is large, the weight approaches $(\nu^2 + 1)N$. When $\nu = \frac{\sqrt{5}-1}{2}$, random algorithm gets a 1.62-approximation, which is a lower bound of one-sided ordinal information setting.

## 3.7 Conclusion

In this chapter we quantified the tradeoffs between the amount of ordinal information available, and the quality of solutions produced by our ordinal approximation algorithms, for metric maximum-weight bipartite matchings. For example, if we are able to collect preference data through surveys, but for each extra preference we must perform a certain extra amount of market research (i.e., increasing $\alpha$ comes at a cost), then our findings would quantify how big we should make $\alpha$ in order to form a good approximation to the best possible matching. All of this is without knowing the true numerical weights, only ordinal information.

One thing to note here is that asking people to list their preference orderings, even partial preference orderings for relatively small $\alpha$, may be prohibitive. Agents are usually willing to name their top 3-10 choices, but not more than that. Notice, however, that all our algorithms can be thought of differently. For example, RSD does not actually require the preference ordering as an input. It simply needs to ask each agent a single question: what is you favorite agent who has not been matched yet? Similarly, our other algorithms can be considered to ask agents a series of questions about their preferences, all of the same form. Such questions (determining their favorite from a set) are usually much easier for agents to answer than the question of specifying a preference ordering.

# CHAPTER 4
# Ordinal Approximation for Social Choice, Matching, and Facility Location Problems Given Candidate Positions

In this chapter we consider general facility location and social choice problems, in which sets of agents $\mathcal{A}$ and facilities $\mathcal{F}$ are located in a metric space, and our goal is to assign agents to facilities (as well as choose which facilities to open) in order to optimize the social cost. We form new algorithms to do this in the presence of only *ordinal information*, i.e., when the true costs or distances of the agents from the facilities are *unknown*, and only the ordinal preferences of the agents for the facilities are available. The main difference between our work and previous work in this area is that, while we assume that only ordinal information about agent preferences is known, we also know the exact locations of the possible facilities $\mathcal{F}$. Due to this extra information about the facilities, we are able to form powerful algorithms which have small *distortion*, i.e., perform almost as well as omniscient algorithms (which know the true numerical distances between agents and facilities) but use only ordinal information about agent preferences. For example, we present natural social choice mechanisms for choosing a single facility to open with distortion of at most 3 for minimizing both the total and the median social cost; this factor is provably the best possible. We analyze many general problems including matching, $k$-center, and $k$-median, and present black-box reductions from omniscient approximation algorithms with approximation factor $\beta$ to ordinal algorithms with approximation factor $1 + 2\beta$; doing this gives new ordinal algorithms for many important problems, and establishes a toolkit for analyzing such problems in the future.

## 4.1 Model and Notation: Social Choice

For the social choice problems studied in this chapter, we let $\mathcal{A} = \{1, 2, \ldots, n\}$ be a set of agents, and let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a set of alternatives, which we will also refer to sometimes as *candidates* or *facilities*. We will typically use $i$ and $j$ to refer to agents and $W, X, Y, Z$ to refer to alternatives. Let $\mathcal{S}$ be the set of total orders on the set of alternatives

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$\mathcal{F}$. Every agent $i \in \mathcal{A}$ has a preference ranking $\sigma \in \mathcal{S}$; by $X \succ_i Y$ we will mean that $X$ is preferred over $Y$ in ranking $\sigma$. Although we assume that each agent has a total order of preference over the alternatives and that this order is known to us, for many of our results it is only necessary that the top choice of each agent is known. We say $X$ is $i$'s top choice if $i$ prefers $X$ to every other alternative in $\mathcal{F}$. We call the vector $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{S}^n$ a preference profile. We say that an alternative $X$ pairwise defeats $Y$ if $|\{i \in \mathcal{A} : X \succ_i Y\}| > \frac{n}{2}$. The goal is to choose a single winning alternative.

**Cardinal Metric Costs.** In this work we take the utilitarian view, and assume that the ordinal preferences $\sigma$ are derived from underlying (latent) cardinal agent costs. Formally, we assume that there exists an arbitrary metric $d : (\mathcal{A} \cup \mathcal{F})^2 \to \mathbb{R}_{\geq 0}$ on the set of agents and alternatives. The cost incurred to agent $i$ by alternative $X$ being selected is represented by $d(i, X)$, which is the distance between $i$ and $X$. Such spatial preferences are relatively common and well-motivated, see for example [23],[31] and the references therein. The underlying distances $d(i, X)$ are *unknown*, but unlike most previous work we *do* assume the distances between *alternatives* are given. The distance between two alternatives $X$ and $Y$ is denoted by $\ell(X, Y)$.

The metric costs $d$ naturally give rise to a preference profile. We say that $d$ is *consistent* with $\sigma$ if $\forall i \in \mathcal{A}$, $\forall X,\ Y \in \mathcal{F}$, if $d(i, X) < d(i, Y)$, then $X \succ_i Y$. It means that the cost of $X$ is less than the cost of $Y$ for agent $i$, so agent $i$ prefers $X$ over $Y$. As described above, we know exactly the distances $\ell$ and the preferences $\sigma$, but do not know the true costs $d$ which give rise to $\sigma$. Let $\mathcal{D}(\sigma, l)$ be the set of metrics that are consistent with $\sigma$ and $\ell$; we know that one of the metrics from this possibly infinite space captures the true costs, but do not know which one. Note that it is proved in [11] that for any preference profile $\sigma$, there exists a metric consistent with $\sigma$, and this conclusion is still true with candidates' ordinal preference profiles for each other, because all the candidates are points on a simplex in the metric construction in [11].

**Social Cost Distortion** We study several objective functions for social cost in this chapter. First, the most common notion of social cost is the sum objective function, defined as $SC_{\sum}(X, \mathcal{A}) = \sum_{i \in \mathcal{A}} d(i, X)$. We also study the median objective function, $SC_{\text{med}}(X, \mathcal{A}) = \text{med}_{i \in \mathcal{A}}(d(i, X))$, as well as the egalitarian objective and many others (see Section 4.2.2).

We use the notion of distortion to quantify the quality of an alternative in the worst case, similarly to the notation in [13],[44]. For any alternative $W$, we define the distortion of $W$ as the ratio between the social cost of $W$ and the optimal alternative:

$$dist_{\sum}(W, \sigma, l) = \sup_{d \in \mathcal{D}(\sigma, l)} \frac{SC_{\sum}(W, \mathcal{A})}{\min_{X \in \mathcal{F}} SC_{\sum}(X, \mathcal{A})} \tag{4.1}$$

$$dist_{\text{med}}(W, \sigma, l) = \sup_{d \in \mathcal{D}(\sigma, l)} \frac{SC_{\text{med}}(W, \mathcal{A})}{\min_{X \in \mathcal{F}} SC_{\text{med}}(X, \mathcal{A})} \tag{4.2}$$

In other words, saying that the distortion of $W$ is at most 3 means that, no matter what the true costs $d$ are (as long as they are consistent with $\sigma$ and $\ell$, which we know), it must be that the social cost of $W$ is within a factor of 3 of the true optimal alternative, which is impossible to compute without knowing the true costs. Because of this, a small distortion value means that there is no need to obtain the true agent costs, and the ordinal information $\sigma$ (together with information $\ell$ about the alternatives) is enough to form a good solution.

A social choice function $f$ on $\mathcal{A}$ and $\mathcal{F}$ takes $\sigma$ and $\ell$ as input, and returns the winning alternative. We say the distortion of $f$ is the same as the distortion of the winning alternative chosen by $f$ on $\sigma$ and $\ell$. In other words, the distortion of a social choice mechanism $f$ on a profile $\sigma$ and facility distances $\ell$ is the worst-case ratio between the social cost of $W = f(\sigma, l)$, and the social cost of the true optimal alternative.

## 4.2 Distortion of Social Choice Mechanisms

### 4.2.1 Distortion of Total Social Cost

In this section, we study the sum objective and provide a deterministic algorithm that gives a distortion of at most 3. According to [23], the lower bound on the distortion for deterministic social choice functions with only the ordinal preferences (without knowing $\ell$) is 3. This occurs in the simple example with 2 alternatives which are tied with approximately half preferring each one. No matter which one is chosen, the true optimum could be the other one, and its social cost can be as much as 3 times better. Because the example in Theorem 4 from [23] only has two alternatives, knowing $\ell$ does not provide any extra information, and thus that example also provides a lower bound of 3 in our setting, although we assume the distances $\ell$ between facilities are known in this chapter. Therefore, our mechanism achieves the best possible distortion in this setting. Note that if we only have ordinal preferences of the agents without the distances between facilities, then the best known approach [25] gave a deterministic algorithm that has a distortion of 4.236. Thus our results establish that by knowing the distances $\ell$ between alternatives, it is possible to reduce the distortion from 4.236 to 3, and no better deterministic mechanism is possible.

**Lemma 4.2.1.** *Let $W$, $X$ be alternatives. If $W \succ_i X$, then $d(i, X) \geq \frac{d(X,W)}{2}$. [Lemma 5 in [23]]*

---

**Algorithm 9:** Algorithm for the minimum total social cost.

---

**Input :** Agents $\mathcal{A} = \{1, 2, \ldots, n\}$,
Alternatives $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$,
Each agent $i$'s top choice alternative,
Distances between alternatives, i.e., $\ell(Y, Z)$, $\forall\, Y, Z \in \mathcal{F}$

**Output:** The winning alternative $W$.

Generate a projected set of agents $\tilde{\mathcal{A}}$: for each agent $i$, create agent $\tilde{i}$ at the location of $i$'s top choice alternative in the metric space, and denote the set of the new agents as $\tilde{\mathcal{A}} = \{\tilde{1}, \tilde{2}, \ldots, \tilde{n}\}$. For each alternative $X \in \mathcal{F}$, calculate the total social cost on $\tilde{\mathcal{A}}$ by choosing $X$, i.e., $SC_{\sum}(X, \tilde{\mathcal{A}}) = \sum_{\tilde{i} \in \tilde{\mathcal{A}}} d(\tilde{i}, X) = \sum_{\tilde{i} \in \tilde{\mathcal{A}}} \ell(\tilde{i}, X)$ .

**Final Output:** Return the alternative $W$ that has the minimum social cost $SC_{\sum}(W, \tilde{\mathcal{A}})$ .

---

In Algorithm 9, we generate a set of projected agents as follows: Given agents $\mathcal{A}$, alternatives $\mathcal{F}$, and the preference profile $\sigma$, for each agent $i$ denote alternative $X_i$ as $i$'s top choice. Then we create a new agent $\tilde{i}$ at the location of $X_i$ in the metric space, as shown in Figure 4.1 (a); consequently, for every $Y \in \mathcal{F}$, $d(\tilde{i}, Y) = d(X_i, Y)$. Denote the set of the new agents as $\tilde{\mathcal{A}} = \{\tilde{1}, \tilde{2}, \ldots, \tilde{n}\}$. For any metric $d$ consistent with $\ell$, $d(\tilde{i}, Y) = d(X_i, Y) = \ell(X_i, Y)$, so the distances between agents in $\tilde{\mathcal{A}}$ and alternatives in $\mathcal{F}$ are known to us, unlike the true distances between $\mathcal{A}$ and $\mathcal{F}$.

**Theorem 4.2.2.** *The distortion of Algorithm 9 for minimum total social cost on $\mathcal{A}$ is at most 3.*

*Proof.* Let $W$ denote the winning alternative. $W$ has the minimum social cost on the agent set $\tilde{\mathcal{A}}$, so for any alternative $Y$, it must be that

$$\frac{SC_{\sum}(W, \tilde{\mathcal{A}})}{SC_{\sum}(Y, \tilde{\mathcal{A}})} = \frac{\sum_{\tilde{i} \in \tilde{\mathcal{A}}} d(\tilde{i}, W)}{\sum_{\tilde{i} \in \tilde{\mathcal{A}}} d(\tilde{i}, Y)} = \frac{\sum_{i \in \mathcal{A}} d(\tilde{i}, W)}{\sum_{i \in \mathcal{A}} d(\tilde{i}, Y)} \leq 1 \tag{4.3}$$

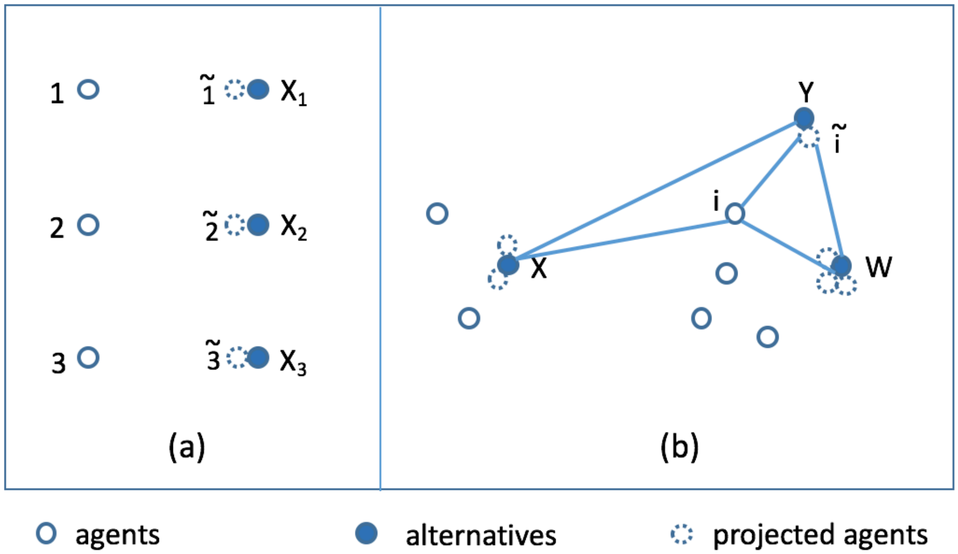


**Figure 4.1: (a) For each agent, generate a projected agent at the location of its top choice alternative. (b) A figure demonstrating agent $i$, $i$'s top choice alternative $Y$, $i$'s projected agent $\tilde{i}$ located at $Y$, the winner $W$, and the optimal alternative $X$ for the proof of Theorem 4.2.2.**

Let $X$ denote the true optimal alternative for $\mathcal{A}$. We want to get $dist_{\sum}(W, \sigma, l)$ by upper bounding the cost incurred by $W$ compared to $X$:

$$\begin{aligned}\frac{SC_{\sum}(W,\mathcal{A})}{SC_{\sum}(X,\mathcal{A})} &= \frac{\sum_{i\in\mathcal{A}} d(i,W)}{\sum_{i\in\mathcal{A}} d(i,X)} \\ &\leq \frac{\sum_{i\in\mathcal{A}}(d(i,\tilde{i}) + d(\tilde{i},W))}{\sum_{i\in\mathcal{A}} d(i,X)} \\ &= \frac{\sum_{i\in\mathcal{A}} d(i,\tilde{i})}{\sum_{i\in\mathcal{A}} d(i,X)} + \frac{\sum_{i\in\mathcal{A}} d(\tilde{i},W)}{\sum_{i\in\mathcal{A}} d(i,X)} \end{aligned} \tag{4.4}$$

The inequality $d(i,W) \leq d(i,\tilde{i}) + d(\tilde{i},W)$ is due to the triangle inequality since $d$ is a metric, as shown in Figure 4.1 (b). $\forall i \in \mathcal{A}$, we know that $\tilde{i}$ is located at $i$'s top choice alternative, so the distance between $i$ and $\tilde{i}$ must be less than (or equal to) the distance between $i$ and any alternative; thus $d(i,\tilde{i}) \leq d(i,X)$. Summing up for all $i \in \mathcal{A}$, we get that $\frac{\sum_{i\in\mathcal{A}} d(i,\tilde{i})}{\sum_{i\in\mathcal{A}} d(i,X)} \leq 1$. For any agent $i$ such that $X$ is not $i$'s top choice, suppose alternative $Y$ is $i$'s top choice, then $\tilde{i}$ has the same location as $Y$ and $d(\tilde{i},X) = d(X,Y)$. By Lemma 4.2.1, $d(i,X) \geq \frac{d(X,Y)}{2}$, thus $d(i,X) \geq \frac{d(\tilde{i},X)}{2}$. For all $i$ that $X$ is $i$'s top choice, $d(\tilde{i},X) = 0$, so the inequality $d(i, X) \geq \frac{d(\tilde{i}, X)}{2}$ holds for all $i \in \mathcal{A}$. Together with inequality 4.3 and 4.4,

$$\frac{SC_{\sum}(W, \mathcal{A})}{SC_{\sum}(X, \mathcal{A})} \leq 1 + \frac{\sum_{i \in \mathcal{A}} d(\tilde{i}, W)}{\sum_{i \in \mathcal{A}} \frac{d(\tilde{i}, X)}{2}} = 1 + 2\frac{\sum_{i \in \mathcal{A}} d(\tilde{i}, W)}{\sum_{i \in \mathcal{A}} d(\tilde{i}, X)} \leq 3$$

□

### 4.2.2 Distortion of Median Social Cost

In this section, we study the median objective function, and provide a deterministic mechanism that gives a distortion of at most 3. Recall that we define the median social cost of an alternative $X$ as $SC_{\text{med}}(X, \mathcal{A}) = \text{med}_{i \in \mathcal{A}}(d(i, X))$. We will refer to this as $\text{med}(X)$ when $d$ and $\mathcal{A}$ are fixed. If $n$ is even, we define the median to be the $(\frac{n}{2} + 1)^{th}$ smallest value of the distances. Note that no deterministic mechanism which only knows ordinal preferences can have worst-case distortion better than 5 (Theorem 25 in [23]). With known distances between candidates, we are able to provide a natural social choice function with distortion of 3, which is also provably the best possible distortion in our setting (consider the example in Theorem 4 from [23] again). Moreover, our social choice function only uses ordinal information about the alternatives, and not the full distances $\ell$; in particular as long as we have ordinal preferences of each alternative for each other alternative (and thus a total order of the distances from each alternative to the others), then our mechanism will work properly. Such ordinal information may be easier to obtain than full distances $\ell$; for example candidates can rank all the other candidates. In particular, given agents with ordinal preferences such that the candidates are a subset of the agents, our mechanism will always form an outcome with small distortion, even if we do not know the distances $\ell$.

Note that using only agents' top choices over alternatives and the distances between the alternatives, as Algorithm 9 does for the total social cost objective, is not enough to give a worst-case distortion of 3 for the median objective. Consider the following example in Figure 4.2: there are 4 alternatives $W, X, Y, Z$, the distances between them are: $d(W, Y) = d(Y, X) = d(X, Z) = d(Z, W) = 2$ and $d(W, X) = d(Y, Z) = 4$. Suppose $W$ is the top choice of agents 1 and 2, $X$ is the top choice of agent 3 and 4, $Y$ is the top choice of agent 5 and 6, and $Z$ is the top choice of agent 7 and 8. This graph is symmetric, so we choose an arbitrary alternative as the winner. Suppose we choose $W$ as the winner, and the distances between agents and candidates are: the distances from agents 1, 2 to $W$ are both 100, the distances from agents 1, 2 to $X, Y, Z$ are all 102. The distances from agents 5, 6 to $Y, X$ are all 1, and the distances from agents 5, 6 to $W, Z$ are all 3. The distances from agents 7, 8 to $Z, X$ are all 1, and the distances from agents 7, 8 to $Y, W$ are all 3. The distances from agents 3, 4 to $X$ are both 1, the distances from 3, 4 to $Y, Z$ are both 3, and the distances from 3, 4 to $W$ are both 5. In this example, the median is the distance from $5^{th}$ closest agent to the winning alternative. $X$ is the optimal alternative with $\text{med}(W) = 1$, while $\text{med}(W) = 5$ has a distortion of 5.

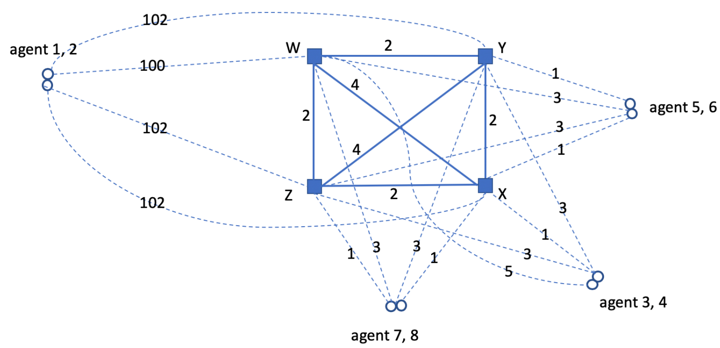


**Figure 4.2: Example that with agents' top choices and the distances between the alternatives, the median objective has a distortion of 5.**

We will use the following Lemmas from [23] in the proof of our algorithm:

**Lemma 4.2.3.** *For any two alternatives $W$ and $Y$, we have $med(W) \leq med(Y) + d(Y, W)$. [Lemma 23 in [23]]*

**Lemma 4.2.4.** *For any two alternatives $Y$ and $P$, if $P$ pairwise defeats (or pairwise ties) $Y$, then $med(Y) \geq \frac{d(Y,P)}{2}$. [Proved in Theorem 24 in [23]]*

**Lemma 4.2.5.** *Let $W, Y$ be alternatives $\in \mathcal{F}$, if $W$ pairwise defeats (or pairwise ties) $Y$, then $med(W) \leq 3med(Y)$. [Proved in Theorem 24 in [23]]*

The main easy insight which we use in the formation of our algorithm comes from the following lemma.

**Lemma 4.2.6.** *For any three alternatives $W$, $Y$, and $P$, if $P$ pairwise defeats (or pairwise ties) $Y$, and $d(Y, W) \leq d(Y, P)$, then $med(W) \leq 3med(Y)$.*

*Proof.* By Lemma 4.2.3, $\text{med}(W) \leq \text{med}(Y) + d(Y, W)$. By Lemma 4.2.4, $\text{med}(Y) \geq \frac{d(Y,P)}{2}$. And we know that $d(Y, P) \geq d(Y, W)$, thus

$$\begin{aligned} \text{med}(W) &\leq \text{med}(Y) + d(Y, W) \\ &\leq \text{med}(Y) + d(Y, P) \\ &\leq \text{med}(Y) + 2\text{med}(Y) \\ &\leq 3\text{med}(Y) \end{aligned}$$

□

We use a natural Condorcet-consistent algorithm to approximate the minimum median social cost with the agents' preference rankings $\sigma$ and the ordinal preferences of every alternative over other alternatives. First, create the majority graph $G = (\mathcal{F}, E)$, i.e., a graph with alternatives as vertices and an edge $(Y, Z) \in E$ if $Y$ pairwise defeats or pairwise ties $Z$. If a Condorcet winner (i.e. an alternative which pairwise defeats all others) exists, then we return it immediately.

Otherwise, we consider each pair of alternatives. By Lemma 4.2.5, if the edge $(W, Y) \in E$, then $\text{med}(W) \leq 3\text{med}(Y)$. When considering an alternative pair $W, Y$, if $(W, Y) \notin E$ and we know that there exists another alternative $P$ which meets the conditions of Lemma 4.2.6, then we add an edge $(W, Y)$ to $G$. It is not difficult to see that whenever the edge $(W, Y)$ is in our graph, this means that $\text{med}(W) \leq 3\text{med}(Y)$. As we prove below, at the end of this process there always exists at least one alternative which has edges to all the other alternatives, and thus the distortion obtained from selecting it is at most 3, no matter which alternative is the true optimal one.

Note that from the ordinal preferences of alternatives over each other, we can get a partial order of distances between the alternatives. Denote this partial order as $\preceq$, i.e., we say that $d(W, Y) \preceq d(W, Z)$ if we know that $W$ prefers $Y$ to $Z$ (we do not have information about strict preference). This is the information we have on hand: we only know the partial order of distances between pairs of alternatives which share an alternative in common. Note, however, that if there exists a cycle in this partial order, i.e., $d(Y_1, Y_2) \preceq d(Y_2, Y_3) \preceq d(Y_3, Y_4) \preceq \cdots \preceq d(Y_k, Y_1) \preceq d(Y_1, Y_2)$, then this implies that all the distances in the cycle are actually equal, and thus we can also add the relations $d(Y_1, Y_2) \succeq d(Y_2, Y_3) \succeq d(Y_3, Y_4) \succeq \cdots \succeq d(Y_k, Y_1) \succeq$

$d(Y_1, Y_2)$. Such cycles are easy to detect (e.g., by forming a graph with a node for every alternative pair and then searching for cycles), and thus we can assume that whenever a cycle exists in our partial order, then for every pair of distances $d(W, Y)$ and $d(W, Z)$ in the cycle, we have both $d(W, Y) \preceq d(W, Z)$ and $d(W, Y) \succeq d(W, Z)$.

**Algorithm 10:** Algorithm for the minimum median social cost.

**Input :** Agents $\mathcal{A} = \{1, 2, \ldots, n\}$,
Alternatives $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$,
The majority graph $G = (\mathcal{F}, E)$,
Ordinal preferences of each alternative over other alternatives,
Partial order of distances between alternatives.

**Output:** The winning alternative $W$.

If there is a Condorcet winner $W$, **return** $W$ **as the winner**.

```
forall alternative pairs W, Y do
    if (W, Y) ∉ E or (Y, W) ∉ E then
        WLOG, suppose (Y, W) exists, but (W, Y) does not exist.
        if there exists an alternative P, such that we have d(Y, W) ⪯ d(Y, P) in
          our partial order information, and P pairwise defeats (or ties) Y then
            Add edge (W, Y) to E;
            continue;
        end
    end
end
```

There must exist an alternative $W$ such that $\forall Y \in \mathcal{F} - \{W\}$, $(W, Y) \in E$. Return $W$ as the winner.

**Lemma 4.2.7.** *Consider the modified majority graph $G = (\mathcal{F}, E)$ at any point during Algorithm 10. For any edge $(W, Y) \in E$, we have that med$(W) \leq 3$med$(Y)$.*

*Proof.* By Lemma 4.2.5, for any edge $(W, Y)$ in the original majority graph, $\text{med}(W) \leq 3\text{med}(Y)$.

Now consider an edge $(W, Y)$ added to $E$ when processing the alternative pair $W, Y$. It must be the case that there exists an alternative $P$, such that $d(Y, W) \leq d(Y, P)$ and $P$ pairwise defeats (or ties) $Y$. By Lemma 4.2.6, $\text{med}(W) \leq 3\text{med}(Y)$. □

**Lemma 4.2.8.** *At the end of Algorithm 10, there must exist an alternative $W$ such that $\forall Y \in \mathcal{F} - \{W\}$, $(W, Y) \in E$.*

*Proof.* We prove this lemma by contradiction. Suppose no such alternative $W$ exists. Then for each alternative $Y$, there is at least one alternative $Z$, such that only $(Z, Y) \in E$ and

$(Y, Z) \notin E$. This is because we start with the majority graph, so at least one edge always exists between every pair. We create another directed graph $G' = (\mathcal{F}, E')$, with $E'$ being all the edges $(Z, Y)$ such that $(Y, Z) \notin E$. Thus any pair of alternatives in $G'$ have at most one direction of edge between them. And by our assumption, each alternative $Y$ has at least one incoming edge in $G'$. Since the in-degree of each node is at least 1 in $G'$, there must be at least one cycle in $G'$. To see this, one can for example take the edge $(Y_2, Y_1)$ coming into $Y_1$, then the edge $(Y_3, Y_2)$ coming into $Y_2$, and proceed in this way until a cycle is formed. Note that every edge in $G'$ must be in the original majority graph, because if we add an edge when processing a pair of alternatives in our algorithm, that pair must have edges in both directions.

Consider a cycle formed by edges $(Y_1, Y_2)$, $(Y_2, Y_3)$, ..., $(Y_{k-1}, Y_k)$, $(Y_k, Y_1)$. When processing the alternative pair $Y_2, Y_3$ in Algorithm 10, we did not add edge $(Y_3, Y_2)$ to $E$, so it must be the case that no alternative $P$ exists such that $d(Y_2, Y_3) \preceq d(Y_2, P)$ and $P$ pairwise defeats (or ties) $Y_2$. But we know that $Y_1$ pairwise defeats (or ties) $Y_2$, because edge $(Y_1, Y_2)$ is in the original majority graph. Then the only possibility is we don't know if $d(Y_2, Y_3) \leq d(Y_1, Y_2)$, i.e., either $d(Y_2, Y_3)$ and $d(Y_1, Y_2)$ are incomparable in our partial order, or we only know that $d(Y_2, Y_3) \succeq d(Y_1, Y_2)$. They cannot be incomparable, since we have the ordinal preferences of $Y_2$ for $Y_1$ and $Y_3$, thus our partial order must state that $d(Y_2, Y_3) \succeq d(Y_1, Y_2)$, i.e., $Y_2$ prefers $Y_1$ to $Y_3$. By the same reasoning, we also get that $Y_3$ prefers $Y_2$ to $Y_4$, and more generally that $Y_i$ prefers $Y_{i-1}$ to $Y_{i+1}$ for all $i$, where $Y_0 = Y_k$ and $Y_{k+1} = Y_1$ since it is a cycle. This means that in our partial order, we have that $d(Y_1, Y_2) \preceq d(Y_2, Y_3) \preceq \cdots \preceq d(Y_{k-1}, Y_k) \preceq d(Y_k, Y_1) \preceq d(Y_1, Y_2)$. Recall, however, that this means we know $d(Y_1, Y_2) = d(Y_2, Y_3) = \cdots = d(Y_{k-1}, Y_k) = d(Y_k, Y_1)$, and before running Algorithm 10, we detect cycles in the partial order of alternative distances, and add the equality information to the partial order. This means that whenever $d(Y_1, Y_2) \preceq d(Y_2, Y_3) \preceq \cdots \preceq d(Y_{k-1}, Y_k) \preceq d(Y_k, Y_1) \preceq d(Y_1, Y_2)$ exists in our partial order, we also have $d(Y_1, Y_2) \succeq d(Y_2, Y_3) \succeq \cdots \succeq d(Y_{k-1}, Y_k) \succeq d(Y_k, Y_1) \succeq d(Y_1, Y_2)$ in the partial order as well. But this gives us a contradiction, since having $d(Y_2, Y_3) \preceq d(Y_1, Y_2)$ in the partial order, combined with the fact that $Y_1$ pairwise defeats $Y_2$, would cause us to add the edge $(Y_3, Y_2)$ in our algorithm, which contradicts the statement that only the edge $(Y_2, Y_3)$ is in the final graph produced by the algorithm, but not $(Y_3, Y_2)$. Thus there must exist at least one alternative with edges from it to all the others. $\square$

**Theorem 4.2.9.** *The distortion of Algorithm 10 for minimum median social cost is at most 3.*

*Proof.* If there is a Condorcet winner, by Lemma 4.2.5, the distortion is at most 3.

Otherwise, by Lemma 4.2.8, the algorithm always returns a winner. Suppose it returns alternative $W$ as the winner, by Lemma 4.2.7, $W$ has a distortion at most 3 with any alternative $X$ as the optimal solution. □

#### 4.2.2.1 Generalizing Median: Percentile Distortion

Instead of just considering the median objective, we also consider a more general objective: the $\alpha$-percentile social cost. Let $\alpha$-PC($Y$) denote the value from the set $\{d(i, Y) : i \in \mathcal{A}\}$, that $\alpha$ fraction of the values lie below $\alpha$-PC($Y$). Thus median is a special case when $\alpha = \frac{1}{2}$, med($Y$) $= \frac{1}{2}$-PC($Y$). It was shown in [23] Theorem 28 that the worst-case distortion when $\alpha \in [0, \frac{1}{2})$ in that setting (only have agent's ordinal preferences over alternatives) is unbounded, and the same example shows $\alpha \in [0, \frac{1}{2})$ in our setting is also unbounded. However, we are able to give a distortion of 3 for $\alpha \in [\frac{1}{2}, 1]$ in this chapter, while for the setting in [23], the lower bound for distortion when $\alpha \in [\frac{1}{2}, \frac{2}{3}]$ is 5. The reason is that the ordinal preferences between alternatives are also available in our setting. We will show that Algorithm 10 gives a distortion of at most 3 not only for the median objective, but also for the general $\alpha$-percentile objective, because all the lemmas we used to prove the conclusion for the median objective could be generalized to $\alpha$-percentile.

We use the following lemma from [23] in the proof of our algorithm:

**Lemma 4.2.10.** *For any two alternatives $W$ and $Y$, we have $\alpha$-PC($W$) $\leq$ $\alpha$-PC($Y$) + $d(Y, W)$. [Lemma 29 in [23]]*

We can generalize Lemma 4.2.6 to the following lemma, and the proof is by using Lemma 4.2.10 instead of Lemma 4.2.3 in the proof of Lemma 4.2.6,

**Lemma 4.2.11.** *For any three alternatives $W$, $Y$, and $P$, if $P$ pairwise defeats (or pairwise ties) $Y$, and $d(Y, W) \leq d(Y, P)$, then $\alpha$-PC($W$) $\leq 3\alpha$-PC($Y$).*

**Theorem 4.2.12.** *The distortion of Algorithm 10 for the $\alpha$-PC objective social cost with $\frac{1}{2} \leq \alpha \leq 1$ is at most 3.*

*Proof.* Note that Lemma 4.2.10 is actually a generalization of Lemma 4.2.3, and Lemma 4.2.11 is a generalization of Lemma 4.2.6. Lemma 4.2.4 generalizes to the $\alpha$-PC objective, because when $\frac{1}{2} \leq \alpha \leq 1$, for any alternative $Y$, we know $\alpha\text{-PC}(Y) \geq \text{med}(Y)$. Lemma 4.2.5 generalizes to the $\alpha$-PC objective. Then Lemma 4.2.7 also generalizes to the $\alpha$-PC objective, because it only uses Lemma 4.2.5 and Lemma 4.2.6 in the proof. And Lemma 4.2.8 still holds for the same algorithm. Thus all the lemmas and properties of the median objective used in the proof of Theorem 4.2.9 could be generalized into the $\alpha$-PC objective, so the conclusion still holds for the $\alpha$-PC objective when $\frac{1}{2} \leq \alpha \leq 1$. □

#### 4.2.2.2 Algorithm 10 and the Total Social Cost

Although Algorithm 10 is designed for the median objective, it also performs quite well for the sum objective. Interestingly, the distortion of this algorithm for the minimum total social cost is at most 5, which slightly worse than the best known deterministic algorithm (with distortion of 4.236) with no knowledge of candidate preferences. Thus this algorithm gives a distortion of 3 for median (and in fact for all $\alpha$-percentile objectives when $\frac{1}{2} \leq \alpha \leq 1$) and distortion of 5 for sum simultaneously. In settings where we are not sure which objectives to optimize, or ones where we care both about the total social good, and about fairness, this social choice mechanism provides the best of both worlds. The lemmas and proofs for this result are similar to Theorem 4.2.9, as follows.

**Lemma 4.2.13.** *Let $W, Y$ be alternatives $\in \mathcal{F}$. If $W$ pairwise defeats (or pairwise ties) $Y$, then $SC_{\sum}(W, \mathcal{A}) \leq 3SC_{\sum}(Y, \mathcal{A})$. [Proved in Corollary 7 in [23]]*

**Lemma 4.2.14.** *For any three alternatives $W$, $Y$, and $P$, if $P$ pairwise defeats (or pairwise ties) $Y$, and $d(Y, W) \leq d(Y, P)$, then $SC_{\sum}(W, \mathcal{A}) \leq 5SC_{\sum}(Y, \mathcal{A})$.*

*Proof.* For all $i \in \mathcal{A}$, we know $d(i, W) \leq d(i, Y) + d(Y, W)$ by the triangle inequality. Summing up for all $i \in \mathcal{A}$, we get $SC_{\sum}(W, \mathcal{A}) \leq SC_{\sum}(Y, \mathcal{A}) + n \cdot d(Y, W)$.

$P$ pairwise defeats (or pairwise ties) $Y$, so at least half of the agents prefer $P$ to $Y$; thus the total social cost of $Y$ is at least the sum of the social cost of these half of agents. By Lemma 4.2.1, we get $SC_{\sum}(Y, \mathcal{A}) \geq \frac{n}{2} \frac{d(Y,P)}{2} = \frac{n}{4} d(Y, P)$. Thus,

$$
\begin{aligned}
SC_{\sum}(W,\mathcal{A}) &\leq SC_{\sum}(Y,\mathcal{A}) + n \cdot d(Y,W) \\
&\leq SC_{\sum}(Y,\mathcal{A}) + n \cdot d(Y,P) \\
&\leq SC_{\sum}(Y,\mathcal{A}) + 4SC_{\sum}(Y,\mathcal{A}) \\
&\leq 5SC_{\sum}(Y,\mathcal{A})
\end{aligned}
$$

□

**Lemma 4.2.15.** *Consider the modified majority graph $G = (\mathcal{F}, E)$ at any point during Algorithm 10. For any edge $(W,Y) \in E$, we have that $SC_{\sum}(W,\mathcal{A}) \leq 5SC_{\sum}(Y,\mathcal{A})$.*

*Proof.* By Lemma 4.2.13, for any edge $(W,Y)$ in the original majority graph, $SC_{\sum}(W,\mathcal{A}) \leq 3SC_{\sum}(W,\mathcal{A})$.

Now consider an edge $(W,Y)$ added to $E$ when processing the alternative pair $W, Y$. It must be the case that there exists an alternative $P$, such that $d(Y,W) \leq d(Y,P)$ and $P$ pairwise defeats (or ties) $Y$. By Lemma 4.2.14, $SC_{\sum}(W,\mathcal{A}) \leq 5SC_{\sum}(Y,\mathcal{A})$. □

**Theorem 4.2.16.** *The distortion of Algorithm 10 for minimum total social cost is at most 5, and this bound is tight.*

*Proof.* If there is a Condorcet winner, by Lemma 4.2.13, the distortion is at most 3. Otherwise, suppose the algorithm returns alternative $W$ as the winner; by Lemma 4.2.15 $W$ has a distortion at most 5 with any alternative $X$ as the optimal solution.

To see that this bound is tight, consider the following example. There are three facilities $W$, $Y$, and $P$. There are $q$ agents who prefer $Y$ to $W$ to $P$, $q$ agents who prefer $P$ to $Y$ to $W$, and 1 agent who prefers $W$ to $P$ to $Y$. We denote these three sets of agents as $\mathcal{A}_Y$, $\mathcal{A}_P$ and $\mathcal{A}_W$ separately. By the preferences of the agents, we know that $Y$ pairwise defeats $W$, $W$ pairwise defeats $P$, and $P$ pairwise defeats $Y$. The distances between facilities are: $d(Y,W) = 2-2\epsilon$, $d(W,P) = 2-\epsilon$, $d(P,Y) = 2$, where $\epsilon$ is a very small positive number. $\mathcal{A}_Y$ is located at the same location as $Y$, so $d(\mathcal{A}_Y,Y) = 0$, $d(\mathcal{A}_Y,P) = 2$, and $d(\mathcal{A}_Y,W) = 2-2\epsilon$. The distances between $\mathcal{A}_P$ and the alternatives are: $d(\mathcal{A}_P,Y) = d(\mathcal{A}_P,P) = 1$, $d(\mathcal{A}_P,W) = 3-2\epsilon$. $\mathcal{A}_W$ has a distance of 1 to all alternatives. Run Algorithm 10 on this example, and consider the alternative pair $W$, $Y$. Because $P$ pairwise defeats $Y$ and $d(Y,W) \preceq d(Y,P)$, we add edge $(W, Y)$ to the graph and make $W$ the winner. The total social cost of $W$ is $q*(2-2\epsilon)+q*(3-2\epsilon)+1 = q(5-4\epsilon)+1$. While the optimal solution is to choose $Y$ as the winner, and get a total social cost of $q+1$. When $q$ is very large and $\epsilon$ is very small, the distortion in this example approaches 5. □

## 4.3 Model and Notation: Facility Assignment Problems

The mechanism we used for approximating the total social cost in Theorem 4.2.2 can be applied to far more general problems. In this section, we describe a set of facility assignment problems that fit in this framework. As before, let $\mathcal{A} = \{1, 2, \dots, n\}$ be a set of agents, and $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ be a set of facilities, with each agent $i$ having a preference ranking $\sigma_i$ over the facilities, and $\sigma = (\sigma_1, \dots, \sigma_n)$.

As in the social choice model, we assume that there exists an arbitrary unknown metric $d : (\mathcal{A} \cup \mathcal{F})^2 \to \mathbb{R}_{\geq 0}$ on the set of agents and facilities. The distances $d(i, F_j)$ between agents and facilities are unknown, but the ordinal preferences $\sigma$ and the distances $\ell$ between facilities are given. Let $\mathcal{D}(\sigma, l)$ be the set of metrics consistent with $\sigma$ and $\ell$, as defined previously in Section 4.1.

Unlike for social choice, our goal is now to choose which facilities to open, and which agents should be assigned to which facilities. Formally, we must choose an assignment $x : \mathcal{A} \to \mathcal{F}$, where $x(i)$ is the facility that $i$ is assigned to. Every $i \in \mathcal{A}$ must be assigned to one (and only one) facility in $\mathcal{F}$; other than that, there could be arbitrary constraints on the assignment. Here are some examples of constraints which fall into our framework: each facility $F_i$ has a capacity, which is the maximum number of agents that can be assigned to $F_i$; at least (or at most) $p$ facilities should have agents assigned to them; agents $i$ and $j$ must be (or must not be) assigned to the same facility, etc. The social choice model is a special case of this one with the constraint that exactly one facility must be opened, and all agents must be assigned to it. Note that the constraints are only on the assignment, and independent of the metric space $d$. An assignment $x$ is valid if it satisfies all constraints. Let $\mathcal{X}$ be the set of all valid assignments.

**The cost function of assignments.** The cost of an assignment $x$ consists of two parts. The first part is the distance cost between agents and facilities. $\forall i \in \mathcal{A}$, let $s_i$ denote the distance between $i$ and the facility it is assigned to, i.e., $s_i = d(i, x(i))$. For a given metric $d$ and assignment $x$, let $s(x,d)$ denote the vector of distances between each $i \in \mathcal{A}$ and $x(i)$, i.e., $s(x,d) = (s_1, s_2, \ldots, s_n)$. Let $c_d : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a cost function that takes a vector of distances as input. For example, this could simply sum up all the distances, take the maximum distance for an egalitarian objective, etc. To be as general as possible, instead of fixing a specific function $c_d$ we consider the set of distance cost functions that are monotonically nondecreasing and subadditive. Formally, $c_d$ is monotonically nondecreasing means that for any vectors $s$ and $s'$ such that $s \leq s'$ componentwise, we have that $c_d(s) \leq c_d(s')$. Any reasonable cost function should satisfy this property if agents desire to be assigned to closer facilities. $c_d$ being subadditive means that for any vectors $s$ and $s'$, we have that $c_d(s+s') \leq c_d(s) + c_d(s')$. While not all functions are subadditive, many important ones are, as they represent the concept of "economies of scale", a common property of realistic costs.

The second part of the assignment cost is the facility cost. Let $c_f(x)$ denote the facility cost for assignment $x$. $c_f$ can be an *arbitrary* function over the assignments, for example, the opening cost of facilities, the penalty (or reward) for assigning certain agents to the same facility, etc. Our framework includes all such functions, and thus is quite general, as we discuss below. The main components needed for our framework to work is that the function $c_f$ does not depend on the distances, only on $x$, and that the function $c_d$ is subadditive.

The total cost $c(x,d)$ of an assignment $x$ is the sum of the distance cost and the facility cost, i.e. $c(x,d) = c_d(s(x,d)) + c_f(x)$. We study algorithms to approximate the minimum cost assignment given only agents' ordinal preferences over facilities, and the distances between facilities, as described above.

**Social Cost Distortion** As for social choice, we use the notion of distortion to measure the quality of an assignment in the worst case, similarly to the notation in [13],[44]. For any assignment $x$, we define the distortion of $x$ as the ratio between the social cost of $x$ and the optimal assignment:

$$dist(x, \sigma, l) = \sup_{d \in \mathcal{D}(\sigma, l)} \frac{c(x,d)}{\min_{x' \in \mathcal{X}} c(x', d)} \tag{4.5}$$

A social choice function $f$ on $\mathcal{A}$ and $\mathcal{F}$ takes $\sigma$ and $\ell$ as input, and returns a valid assignment on $\mathcal{A}$ and $\mathcal{F}$. We say the distortion of $f$ on $\sigma$ and $\ell$ is the same as the distortion of the assignment returned by $f$. In other words, the distortion of an assignment function $f$ on a profile $\sigma$ and facility distances $\ell$ is the worst-case ratio between the social cost of

$x = f(\sigma, l)$, and the social cost of the true optimal assignment, to obtain which we would need the true distances $d$.

**Approximation ratio of omniscient algorithms** Consider omniscient algorithms which know the true numerical distances between agents and facilities for the facility assignment problems, in other words, the metric $d$. In some sense, the goal of our work is to determine when algorithms with only limited information can compete with such omniscient algorithms. With the full distances information, we can of course obtain the optimal assignment using brute force, while for our algorithms with limited knowledge this is impossible even given unlimited computational resources. Nevertheless, we are also interested in what is possible to achieve if we restrict ourselves to polynomial time. To differentiate traditional approximation algorithms from algorithms with small distortion, suppose that an omniscient approximation algorithm $\tilde{f}$ returns assignment $x$. Then we denote the approximation ratio of a valid assignment $x$ as:

$$ratio(x) = \frac{c(x, d)}{\min_{x' \in \mathcal{X}} c(x', d)} \tag{4.6}$$

Thus we say the approximation ratio of an omniscient algorithm $\tilde{f}$ is at most $\beta$ if for any input of the problem, the assignment $x$ returned by $\tilde{f}$ has $ratio(x) \leq \beta$.

### 4.3.1 Examples of Facility Assignment Problems

In this section we illustrate that our framework is quite general by giving various important examples which fit into our framework. In the section which follows, we prove a general black-box reduction theorem for our framework, and thus immediately obtain mechanisms with small distortion for all these examples simultaneously.

The total social cost problem we discussed in Section 4.2.1 is a special case of the facility assignment problem such that the constraint is only one facility (alternative) is chosen, and all agents are assigned to it. For any assignment $x$, the facility cost function $c_f(x) = 0$, and the distance cost function $c_d(s(x, d))$ is the sum of distances from the winning alternative to all agents in the metric $d$. $c_d$ is monotone and additive (thus subadditive). Here are some other examples that fit in our framework:

**Minimum weight metric bipartite matching.** Given a set of agents $\mathcal{A}$ and a set of facilities $\mathcal{F}$ such that $|\mathcal{A}| = |\mathcal{F}| = n$. $G = (\mathcal{A}, \mathcal{F}, E)$ is an undirected complete bipartite graph. The facilities and agents lie in a metric space $d$. The weight of each edge $(i, F) \in E$ is the distance between $i$ and $F$, $w(i, F) = d(i, F)$. The goal is to find a minimum weight perfect matching of the bipartite graph given only ordinal information. This setting has been studied before, and the best distortion bound known is $n$ [5] given by RSD for the case when only the ordinal preferences $\sigma$ are known. Our results show that if we also know the distances $\ell$ between facilities, then even without knowing the distances $d$ between agents and facilities, it is possible to create simple mechanisms with distortion at most 3 (we can show that no better bound is possible for this setting). Thus having a bit more information about the facilities immediately improves the distortion bound by a very large amount. We show this result by using our facility assignment framework above: the constraint here is that each facility has a capacity of 1, thus a valid assignment is a perfect matching of the bipartite graph. For any assignment $x$, the facility cost function is $c_f(x) = 0$, and the distance cost function $c_d(s(x, d))$ is the total edge weight in the assignment. $c_d$ is monotone and additive (thus subadditive).

**Egalitarian bipartite matching.** With the same bipartite graph as in minimum weight matching problems, the only difference is that the goal of egalitarian bipartite matching is to find a perfect matching such that maximum edge weight (instead of the total weight) in the matching is minimized [90].

The egalitarian bipartite matching problem is the same as minimum weight bipartite matching except the distance cost function $c_d(s(x, d))$ is the maximum edge weight in the assignment. This function is also monotone and subadditive.

**Metric Facility Location.** In this problem, one is given a set of agents $\mathcal{A}$ and a set of facilities $\mathcal{F}$ such that $|\mathcal{A}| = n$, $|\mathcal{F}| = m$. The facilities and agents lie in a metric space $d$. Each facility $F_j \in \mathcal{F}$ has an opening cost $f_j$. Each agent is assigned to a facility; in different versions there may be capacities on the number of agents assigned to a facility, lower bounds on the number of agents assigned to a facility, or various other constraints [91]. The goal is to find a subset of facilities $\hat{\mathcal{F}} \subseteq \mathcal{F}$ to open, so that the sum of opening costs for facilities in $\hat{\mathcal{F}}$ and total distance of the assignment is minimized.

Our framework allows arbitrary constraints on what constitutes a valid assignment, which captures facilities with capacities or lower bounds if needed. For any assignment $x$, the facility cost function $c_f(x)$ is the sum of the opening costs $f_j$ for those facilities $F_j$ that have at least one agent assigned to it. The distance cost function $c_d(s(x, d))$ is the total distances in the assignment, which is monotone increasing and additive (thus subadditive).

**$k$-center problem.** The goal in this classic problem is to open a set of $k$ facilities, with each agent assigned to the closest one. The optimal solution is the subset of $\hat{\mathcal{F}}$ which minimizes $\max_{i\in\mathcal{A}} d(i, x(i))$. To express this in our framework, the constraint is that no more than $k$ facilities have agents assigned to them. For any assignment $x$, the facility cost function $c_f(x) = 0$, and the distance cost function $c_d(s(x, d))$ is the maximum distance between any agent and facility in the assignment.

**$k$-median problem.** This classic problem is the same as $k$-center, except the goal is to minimize the sum of distances of agents to the facilities instead of the maximum distance.

## 4.4 Distortion of Facility Assignment Problems

In this section, we study general facility assignment problems, as described in Section 4.3, and form mechanisms with small distortion. First, we construct a projected problem such that the distances between agents and facilities are known, so it could be solved by an omniscient algorithm. Then we map the result of the projected problem to the original problem and bound the distortion of the original problem.

Given agents $\mathcal{A} = \{1, 2, \ldots, n\}$ and facilities $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$, suppose facility $F'$ is $i$'s top choice in $\mathcal{F}$. We create a new agent $\tilde{i}$ at the location of $F'$ in the metric space. Consequently, $\forall F \in \mathcal{F}$, $d(\tilde{i}, F) = d(F', F)$. Denote the set of the new agents as $\tilde{\mathcal{A}} = \{\tilde{1}, \tilde{2}, \ldots, \tilde{n}\}$.

The original assignment problem is on agents $\mathcal{A}$ and facilities $\mathcal{F}$, and only the ordinal preferences of the agents in $\mathcal{A}$ over the facilities are given. The projected problem is on agents $\tilde{\mathcal{A}}$ and facilities $\mathcal{F}$, and we know the actual distances between agents in $\tilde{\mathcal{A}}$ and facilities $\mathcal{F}$, since we know the distances $\ell$ between facilities. The constraints and costs $c_d$ and $c_f$ remain the same for both the original and the projected problem; the only difference is in the distances $d$. Our main result is that if we have a $\beta$-approximation assignment to the minimum assignment cost on the projected problem, then we can get an assignment that has a distortion of $2\beta + 1$ for the original problem in polynomial time.

**Theorem 4.4.1.** *Given a valid assignment $\tilde{x}$ for the projected problem on $\tilde{\mathcal{A}}$ and $\mathcal{F}$, with ratio$(\tilde{x}) \leq \beta$, the assignment $x(i) = \tilde{x}(\tilde{i})$ has distortion at most $(1 + 2\beta)$ for the original assignment problem on $\mathcal{A}$ and $\mathcal{F}$.*

*Proof.* First, $\tilde{x}$ is a valid assignment for the projected problem on $\tilde{\mathcal{A}}$ and $\mathcal{F}$, so $x$ must also be a valid assignment for the original problem on $\mathcal{A}$ and $\mathcal{F}$. This is because the constraints are only on the assignment, and are independent of the metric space $d$. For the same reason, the facility cost of $x$ equals the facility cost of $\tilde{x}$, $c_f(x) = c_f(\tilde{x})$.

Now consider the distance cost of $x$. Let $x^*$ denote the optimal assignment for the original problem. $\forall i \in \mathcal{A}$, let $s_i = d(i, x(i))$, $t_i = d(i, \tilde{i})$, $b_i = d(\tilde{i}, x(i))$. Similarly, let $s_i^* = d(i, x^*(i))$, $b_i^* = d(\tilde{i}, x^*(i))$.

For any agent $i$ and facility $x(i)$, by triangle inequality,

$$s_i = d(i, x(i)) \leq d(i, \tilde{i}) + d(\tilde{i}, x(i)) = t_i + b_i$$

Because $c_d$ is monotonically nondecreasing and subadditive,

$$\begin{aligned} c_d(s_1, s_2, \ldots, s_n) &\leq c_d(t_1 + b_1, t_2 + b_2, \ldots, t_n + b_n) \\ &\leq c_d(t_1, t_2, \ldots, t_n) + c_d(b_1, b_2, \ldots, b_n) \end{aligned}$$

Therefore, the cost of our assignment $x$ is bounded as follows:

$$\begin{aligned} c_f(x) + c_d(s(x, d)) &= c_f(x) + c_d(s_1, s_2, \ldots, s_n) \\ &= c_f(\tilde{x}) + c_d(s_1, s_2, \ldots, s_n) \\ &\leq c_f(\tilde{x}) + c_d(t_1, t_2, \ldots, t_n) + c_d(b_1, b_2, \ldots, b_n) \end{aligned}$$

Because $\tilde{i}$ is located at $i$'s top choice facility, and $x^*(i)$ is a facility, we thus know that $t_i \leq s_i^*$, and by monotonicity $c_d(t_1, t_2, \ldots, t_n) \leq c_d(s_1^*, s_2^*, \ldots, s_n^*)$. Thus,

$$\begin{aligned} c_f(x) + c_d(s(x, d)) &\leq c_f(\tilde{x}) + c_d(t_1, t_2, \ldots, t_n) + c_d(b_1, b_2, \ldots, b_n) \\ &\leq c_f(\tilde{x}) + c_d(s_1^*, s_2^*, \ldots, s_n^*) + c_d(b_1, b_2, \ldots, b_n) \end{aligned}$$

We know that $\tilde{x}$ is a $\beta$-approximation of the optimal assignment for the projected problem. Its total cost is exactly $c_f(\tilde{x}) + c_d(b_1, b_2, \ldots, b_n)$, since the distance from $\tilde{i}$ to

$\tilde{x}(\tilde{i}) = x(i)$ is exactly $b_i$. Now consider another assignment for the projected problem, in which $\tilde{i}$ is assigned to $x^*(i)$. The cost of this assignment is $c_f(x^*) + c_d(b_1^*, b_2^*, \ldots, b_n^*)$, by definition of $b_i^*$. Since $\tilde{x}$ is a $\beta$-approximation, we therefore know that

$$c_f(\tilde{x}) + c_d(b_1, b_2, \ldots, b_n) \leq \beta c_f(x^*) + \beta c_d(b_1^*, b_2^*, \ldots, b_n^*),$$

and thus

$$\begin{aligned} c_f(x) + c_d(s(x,d)) &\leq c_f(\tilde{x}) + c_d(s_1^*, s_2^*, \ldots, s_n^*) + c_d(b_1, b_2, \ldots, b_n) \\ &\leq c_d(s_1^*, s_2^*, \ldots, s_n^*) + \beta c_f(x^*) + \beta c_d(b_1^*, b_2^*, \ldots, b_n^*) \end{aligned}$$

For any agent $i$ and facility $x^*(i)$ in $x^*$, by triangle inequality,

$$b_i^* = d(\tilde{i}, x^*(i)) \leq d(i, x^*(i)) + d(i, \tilde{i}) \leq 2d(i, x^*(i)) = 2s_i^*$$

$d(i, \tilde{i}) \leq d(i, x^*(i))$ above since $\tilde{i}$ is located at the closest facility to $i$. Because $c_d$ is monotone and subadditive, we also have that

$$c_d(b_1^*, b_2^*, \ldots, b_n^*) \leq c_d(2s_1^*, 2s_2^*, \ldots, 2s_n^*) \leq 2c_d(s_1^*, s_2^*, \ldots, s_n^*)$$

Putting everything together,

$$\begin{aligned} c_f(x) + c_d(s(x,d)) &\leq c_d(s_1^*, s_2^*, \ldots, s_n^*) + \beta c_f(x^*) + \beta c_d(b_1^*, b_2^*, \ldots, b_n^*) \\ &\leq \beta c_f(x^*) + c_d(s_1^*, s_2^*, \ldots, s_n^*) + 2\beta c_d(s_1^*, s_2^*, \ldots, s_n^*) \\ &= \beta c_f(x^*) + (1 + 2\beta) c_d(s_1^*, s_2^*, \ldots, s_n^*) \\ &\leq (1 + 2\beta)(c_f(x^*) + c_d(s(x^*, d))) \end{aligned}$$

□

Note that the above theorem immediately implies that if we are only concerned with what is possible to achieve given limited ordinal information in addition to distances between facilities, and are not worried about our algorithms running in polynomial time, then we can always form an assignment with distortion of at most 3 from knowing only $\sigma$ and $\ell$. This is because we can solve the projected problem with brute force, and then we have $\beta = 1$. This bound of 3 is tight for many facility assignment problems: consider for example an instance of min-cost metric matching with two agents and two facilities, with both preferring $F_1$ to $F_2$. One of the agents has distance to $F_1$ of 0, and one is located halfway between $F_1$ and $F_2$, but since we only have ordinal information we do not know which agent is which. If we assign the wrong agent to $F_1$, then we end up with distortion of 3, and it is impossible to do better for any deterministic mechanism.

If on the other hand we want to form poly-time algorithms with small distortion, the above theorem gives a black-box reduction: if we have a $\beta$-approximation algorithm for the omniscient case, then we can form a $1 + 2\beta$-distortion algorithm for the ordinal case. Actually, we get a $1 + 2\beta$-distortion for the distance cost, and a $\beta$-distortion for the facility cost, which is shown in the second-to-last line of the proof for Theorem 4.4.1. This leads to the following corollaries:

**Corollary 4.4.1.1.** *We can achieve the following distortion in polynomial time:*

1. *At most 3 for the minimum weight bipartite matching problem.*
2. *At most 3 for Egalitarian bipartite matching.*
3. *At most 3.976 for the facility location problem (1.488-approximation for the facility cost, and 3.976-approximation for the distance cost).*
4. *At most 5 for the k-center problem.*
5. *At most 6.35 for the k-median problem.*

*Proof.* Min-cost matching and egalitarian matching are poly-time solvable, so $\beta = 1$. For the latter, one can fix the threshold weight $t$ such that every edge chosen should be at most $t$, and then determine if such a matching exists. Performing a binary search on $t$ gives an efficient algorithm. For facility location, one can use the omniscient algorithm which is a

1.488-approximation in [26]. For the $k$-center problem, a greedy algorithm [27] gives a 2-approximation for the setting that agents are a subset of facilities, which is the case in our projected problem. [28] gives a 2.675-approximation omniscient algorithm for the $k$-median problem when agents are a subset of facilities, thus it also gives a 2.675-approximation for our projected problem. □

Note that the median function, unlike sum and maximum, is not subadditive, and thus does not fit into our framework. In fact, while both min-cost and egalitarian matching problems have algorithms with small distortion in our setting, the same is not possible for forming a matching where the objective function is the cost of the *median* edge.

## 4.5 Bad Examples and Lower Bounds

Note that our Algorithm 10 is only for social choice problems, and does not fit in the definition of our general facility assignment problems. This is because the median cost function, unlike sum and maximum, is not subadditive. In fact, while both min-cost and egalitarian matching problems have algorithms with small distortion in our setting, the same is not possible for forming a matching where the objective function is the cost of the *median* edge.

**Theorem 4.5.1.** *The worst-case distortion of the median-cost bipartite matching problem in a metric space (given both agent preference profiles and distances between facilities) is unbounded.*

*Proof.* Consider the following example: there are three agents $a$, $b$, $c$, and three facilities $X$, $Y$, $Z$. The preferences of agents are: $a, b \in XYZ$, while $c \in ZXY$. The distances between facilities are: $l(X,Y) = 2$, $l(X,Z) = l(Y,Z) = 1000$. The distances between the agents and facilities are, of course, unknown. Consider the instance $d(c,Z) = \epsilon$, $d(a,X) = 2\epsilon$, and $d(b,X) = d(b,Y) = 1$. $\epsilon$ is a very small positive real number, and other distances not given obey triangle inequality. In this instance, the optimal solution is $x^* = \{(a,X),(b,Y),(c,Z)\}$, which gives a median value of $2\epsilon$. But because $a$ and $b$ have the same preference profile, the instance could also be $d(c,Z) = \epsilon$, $d(b,X) = 2\epsilon$, and $d(a,X) = d(a,Y) = 1$. If we still return the assignment $x^*$ for this instance, the median would be 1. The distortion is arbitrarily bad when $\epsilon$ approaches 0. □

The following Theorems show some of the lower bounds mentioned in Table 1.2.

**Theorem 4.5.2.** *The worst-case distortion for the facility location problem in a metric space (given only agents' preference profiles) is unbounded.*

*Proof.* Consider the following example: there are two agents 1, 2, and two facilities $X$, $Y$. Agent 1 prefers $X$ to $Y$, while agent 2 prefers $Y$ to $X$. The opening costs are: $c_f(X) = 1$, $c_f(Y) = 100$. We can choose to open one facility or both of them.

**Case 1.** Suppose we only open $X$. Consider the following distances between the agents and facilities: $d(1, X) = d(2, Y) = 1$, $d(1, Y) = d(2, X) = L$, for some very large $L$. If we only open $X$, then the total cost is $> L$. While the optimal solution is to open both $X$ and $Y$, which has a total cost of 103. The distortion is unbounded.

**Case 2.** Suppose we only open $Y$. Consider the same distances as in **Case 1**, then the total cost is also $L$. And the optimal solution still has a total cost of 103. The distortion is unbounded.

**Case 3.** Suppose we open both facilities. Consider the following distances between the agents and facilities: $d(1, X) = d(1, Y) = d(2, X) = d(2, Y) = \epsilon$, where $\epsilon$ is a very small positive real number. If we open both facilities, the total cost is $101 + 2\epsilon$. While the optimal solution is to only open $X$ , which has a total cost of $1 + 2\epsilon$. If we increase $c_f(Y)$, the approximation ratio is unbounded. □

**Theorem 4.5.3.** *The worst-case distortion for the k-median problem in a metric space (given only agents' preference profiles) is at least $\Omega(n)$.*

*Proof.* Consider the following example: There are three facilities $X$, $Y$, and $Z$. There are $q$ agents who prefer $X$ to $Y$ to $Z$, $q$ agents who prefer $Y$ to $X$ to $Z$, and 1 agent who prefers $Z$ to $X$ to $Y$. We denote these three sets of agents as $\mathcal{A}_X$, $\mathcal{A}_Y$ and $\mathcal{A}_Z$ separately. Suppose $k = 2$, then we have three choices of the winners:

**Case 1.** Choose $X, Y$ as the winners. Consider the following distances between agents and facilities: $d(X, Y) = 1$, $d(Y, Z) = d(X, Z) = L$ for some very large $L$. $\mathcal{A}_X$ is located at the same location as $X$, $\mathcal{A}_Y$ is at the same location as $Y$, and $\mathcal{A}_Z$ is at the same location as $Z$. The cost of choosing $X, Y$ as the winners is $L$ because we need to assign the agent in $\mathcal{A}_Z$ to $X$ or $Y$. While the optimal solution is to choose $Y, Z$ as the winners, and get a total cost of $q$. So the distortion in this case is unbounded.

**Case 2.** Choose $X, Z$ as the winners. Consider the following distances between agents and facilities: $d(X, Y) = d(Y, Z) = d(X, Z) = 1$, and $\mathcal{A}_X$ locate on top of $X$, $\mathcal{A}_Y$ locate on top of $Y$, and $\mathcal{A}_Z$ locate on top of $Z$. The cost of choosing $X, Z$ as the winner is $q$, while the optimal solution is to choose $X, Y$ as the winners, and get a total cost of 1. The distortion is $q$ in this case.

**Case 3.** Choose $Y, Z$ as the winners. Consider the same distances as in **Case 2**. If we choose $Y, Z$ as the winners, the total cost is still $q$, and the distortion of this case is also $q$.

The total number of agents is $n = 2q + 1$, so we can conclude that the distortions in all these three cases are at least $\Omega(n)$. □

**Theorem 4.5.4.** *The worst-case distortion of the egalitarian bipartite matching problem in a metric space (given only agents' preference profiles) is at least* 2.

*Proof.* Consider the following example: there are two agents 1, 2, and two facilities $X$, $Y$. Both agents prefer $X$ to $Y$. W.L.O.G., assume we match agent 1 to $X$, and agent 2 to $Y$. Suppose the distances between agents and facilities are: $d(1, X) = d(1, Y) = 1$, $d(2, X) = \epsilon$, $d(2, Y) = 2$, where $\epsilon$ is a very small positive real number. The egalitarian cost of our matching is 2, while the optimal solution is to match agent 1 to $Y$, and agent 2 to $X$, which has a cost of 1. □

## 4.6 Conclusion

In this chapter, we provided two mechanisms to solve different social cost problems. The first one makes use of the distances between facilities and an omniscient algorithm to get a low distortion for general facility assignment problems. The second mechanism is a new voting rule for social choice which simultaneously achieves a distortion of 3 for many objectives, including the cost of the median voter, and a distortion of 5 for the total social cost at the same time. The first mechanism requires the full distances $\ell$, but only needs the top choice from each agent. Thus, it puts only a small load on the agents which submit their preferences, but requires the mechanism designer to collect more information about the facilities and their distances to each other. The second mechanism, on the other hand, only requires ordinal preference information from the facilities, but needs the full preference ranking from the agents instead of just the top choice. It is especially appropriate for settings in which the candidates or alternatives are agents themselves.

# CHAPTER 5
# Awareness of Voter Passion Greatly Improves the Distortion of Metric Social Choice

In this chapter, we develop new voting mechanisms for the case where voters and candidates are located in an arbitrary unknown metric space, and the goal is to choose a candidate minimizing social cost: the total distance of the voters to this candidate. Previous work has often assumed that only the ordinal preferences of the voters are known (instead of their true costs), and focused on minimizing distortion: the quality of the chosen candidate as compared to the best possible candidate. In chapter, we instead assume that a (very small) amount of information is known about the voter preference *strengths*, not just about their ordinal preferences. We provide mechanisms with much better distortion when this extra information is known as compared to mechanisms which use only ordinal information. We quantify tradeoffs between the amount of information known about preference strengths and the achievable distortion. We further provide advice about which type of information about preference strengths seems to be the most useful.

## 5.1 Model and Notation

As in previous work on metric distortion, we have a set of voters $\mathcal{V} = \{1, 2, \dots, n\}$ and a set of candidates (or alternatives) $\mathcal{C}$. These voters and candidates correspond to points in an arbitrary (unknown) metric space $d$. The voter preferences over the candidates are induced by the underlying metric, i.e., voters prefer candidates who are closer to them. Voter $i$ prefers candidate $P$ over candidate $Q$ (i.e., $P \succ_i Q$) only if $d(i, P) \leq d(i, Q)$. Moreover, we assume that the strengths of voter preferences are induced by these latent distances. If $i$ prefers $P$ over $Q$, then the strength of this preference is $\alpha_i^{PQ} = \frac{d(i,Q)}{d(i,P)}$. The cost to voter $i$ if candidate $P$ is elected is $d(i, P)$, and the goal is to select the candidate minimizing the Social Cost: $SC(P) = \sum_{i \in \mathcal{V}} d(i, P)$.

Given a set of preference strength thresholds $\{1 \leq \tau_1 < \tau_2 < \ldots < \tau_m\}$, voters report

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the largest threshold which their preference strength exceeds for each pair of candidates. We let $A_l^{PQ} = \{i \in \mathcal{V} : d(i,P) \leq d(i,Q)$ and $\tau_l \leq \alpha_i^{PQ} < \tau_{l+1}\}$ and $B_l^{PQ} = \{j \in \mathcal{V} : d(j,Q) \leq d(j,P)$ and $\tau_l \leq \alpha_j^{QP} < \tau_{l+1}\}$. When $\tau_1 = 1$ we know the preferred candidate of every voter, i.e., for voter $i$ and each pair of candidates $P$ and $Q$, we know whether $i$ prefer $P$ or $Q$. When $\tau_1 > 1$ we let $C$ denote the set of voters with preference strength strictly less than $\tau_1$ whose preferred candidate is unknown. When $m \to \infty$, we know the exact preference strength of every voter for every pair of candidates. For convenience in expressing some of our bounds, we also sometimes say $\tau_{m+1} = \infty$ and $\tau_0 = 1/\tau_1$.

In previous work on metric distortion only the ordinal preferences were known, i.e., whether $(P \succ_i Q)$ or $(Q \succ_i P)$. In this chapter, however, we assume that we are also given some information about the preference strengths $\alpha_i^{PQ} = \frac{d(i,Q)}{d(i,P)}$ as well. Note that knowing these values still does not tell us how $d(i,P)$ compares with $d(j,P)$ for $i \neq j$, only how strongly each voter feels when comparing different candidates.

For a given voting rule $\mathcal{R}$ and instance $I = \{\mathcal{V}, \mathcal{C}, d\}$, let $P_I$ be the winning candidate selected by $\mathcal{R}$ and let $Z_I$ be the best available candidate (the one minimizing the Social Cost). Then, the *distortion of winning candidate* $P_I$ is defined as

$$\delta_I = \frac{SC(P_I)}{SC(Z_I)} \tag{5.1}$$

The *distortion of a voting rule* $\mathcal{R}$ is defined by its behavior on a worst-case instance:

$$\delta = \max_I \delta_I = \max_I \frac{SC(P_I)}{SC(Z_I)} \tag{5.2}$$

### 5.1.1 Lower Bounds on Distortion with Preference Strengths

Here, we provide lower bounds on the minimum distortion any deterministic mechanism can achieve given only preference strength information. First, note that even if all *exact* preference strengths were known to us, we still would not be able to choose the optimal candidate: knowing the relative strength of preference for every voter is not the same thing as knowing their exact distances to every candidate (i.e., we would only know $\alpha_i = \frac{d(i,P)}{d(i,Q)}$ and not $d(i,P)$ and $d(i,Q)$ themselves).

**Theorem 5.1.1.** *No deterministic mechanism with only preference strength information can achieve a worst-case distortion less than* $\sqrt{2}$.

*Proof.* The example used is in 1D, where candidates $P$ and $Q$ are represented by points on a line. We normalize the distances so that $P$ is at location 0 and $Q$ is at location 1. Suppose half the voters prefer $P$ with strength $1+\sqrt{2}$, and the other half prefer $Q$ with strength $1+\sqrt{2}$. Since this is the only information known to the mechanism, the mechanism must tie-break in some arbitrary way (if tie-breaking is undesirable, we can have one extra voter prefer $P$, which will result in distortion arbitrarily close to $\sqrt{2}$ instead of exactly $\sqrt{2}$). Thus without loss of generality, we let $P$ be the winner over $Q$.

Suppose the true location of the voters is as follows. Half of the voters are located at $\frac{1}{2+\sqrt{2}}$ and the other half are located at $\frac{3+2\sqrt{2}}{2+\sqrt{2}}$. All voters have a preference strength of $1+\sqrt{2}$. If there are $N$ voters, the candidates have social costs $SC(P) = 2N$ and $SC(Q) = \sqrt{2}N$. Thus, if $P$ wins we have a lower bound on distortion of $\delta \geq \sqrt{2}$. □

Of course it is unrealistic to expect to know the exact preference strengths of all voters. Below we give a general lower bound for the best distortion possible given knowledge of certain preference thresholds.

**Theorem 5.1.2.** *When given knowledge of $m$ fixed thresholds, no deterministic mechanism can always achieve a distortion less than* $\max_{0\leq l\leq m}\{\frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1}\}$

*Proof.* The proof follows from the following 3 lemmas. The examples used for these lemmas are all in 1D, where candidates $P$ and $Q$ are represented by points on a line. We normalize the distances so that $P$ is at location 0 and $Q$ is at location 1 and use $\epsilon$ to denote an infinitesimal quantity. Without loss of generality, we let $P$ be the winner over $Q$. Recall that we have defined $\tau_0 = \frac{1}{\tau_1}$ and $\tau_{m+1} = \infty$ for convenience.

**Lemma 5.1.3.** *If we have a set of thresholds of which the smallest is $\tau_1 > 1$, no deterministic mechanism can always achieve a distortion less than $\tau_1$.*

*Proof.* Suppose all $N$ voters are located at position $\frac{\tau_1}{\tau_1+1} - \epsilon$. All voters therefore have preference strength less than $\tau_1$, so $|C| = N$ and the preferred candidates of the voters are unknown. If $P$ wins over $Q$ due to tie-breaking, as $\epsilon \to 0$ this yields a lower bound on distortion of $\delta \geq \tau_1$. □

**Lemma 5.1.4.** *If we have a set of thresholds of which the largest is $\tau_m > 1$, no deterministic mechanism can always achieve a distortion less than $\frac{\tau_m+2}{\tau_m}$.*

*Proof.* Suppose half of the voters are located on top of $Q$ at position 1 and the other half of voters are located at $\frac{1}{\tau_m+1} - \epsilon$, so $|A_m| = |B_m| = N/2$. If $P$ wins over $Q$ due to tie-breaking, as $\epsilon \to 0$ this yields a lower bound on distortion of $\delta \geq \frac{\tau_m+2}{\tau_m}$. □

**Lemma 5.1.5.** *If we have a set of thresholds, of which two consecutive thresholds are $\tau_l$ and $\tau_{l+1}$ where $\tau_l < \tau_{l+1}$, no deterministic mechanism can achieve a distortion less than $\frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1}$.*

*Proof.* Suppose half of the voters are located at position $\frac{1}{\tau_l+1} - \epsilon$ and the other half are located at position $1 + \frac{1}{\tau_{l+1}-1} + \epsilon$. Once again, the mechanism must choose randomly between the candidates because $|A_l| = |B_l| = N/2$. Therefore, if $P$ wins over $Q$ due to tie-breaking, as $\epsilon \to 0$, this yields a lower bound on distortion of $\delta \geq \frac{\tau_l\tau_{l+1}+2\tau_{l+1}-1}{\tau_l\tau_{l+1}+1}$. □

The combination of the three preceding lemmas guarantees the lower bound of Theorem 5.1.2. □

## 5.2 Adding the Knowledge of a Single Threshold $\tau$ to Ordinal Preferences

### 5.2.1 Distortion with Two Candidates

In this section we begin by analyzing the case with only two possible candidates. In the section that follows, we use these results to form mechanisms with small distortion for multiple candidates. Suppose there are two candidates $P$ and $Q$. We are given the voters' ordinal preferences, and a strength threshold $\tau$, i.e., for every voter we only know two bits of information: whether they prefer $P$ or $Q$, and whether their preference is strong ($> \tau$) or weak ($\leq \tau$). Note that our results still hold if we only have this knowledge in aggregate, i.e., if for both $P$ and $Q$ we know approximately how many people prefer $P$ to $Q$ strongly versus weakly, and vice versa.

Notice that preference strengths tell us little about the true underlying distances for voters with weak preference strengths, because the preference strength of a voter almost directly between $P$ and $Q$ who is very close to both can have the same preference strength as a voter who is very distant from both candidates. However, if a voter's preference strength is large, we know they must be fairly close to one of the candidates - and it is these passionate voters who contribute most to distortion.

**Weighted Majority Rule 1.** *Given voters' preferences and a threshold* $\tau$ *for two candidates, if* $\tau \geq \sqrt{2}+1$*, assign weight* $\frac{\tau+1}{\tau-1}$ *to all the voters with preference strengths* $> \tau$ *and weight* 1 *to all the voters with preference strengths* $\leq \tau$*. If* $\tau < \sqrt{2}+1$*, assign weight* $\tau$ *to all the voters with preference strengths* $> \tau$ *and weight* 1 *to all the voters with preference strengths* $\leq \tau$*. Choose the candidate by a weighted majority vote.*

The following theorem shows that the above voting rule produces much better distortion than anything possible from knowing only the ordinal preferences. Moreover, due to the lower bounds in the previous section, this is the best distortion possible (apply Theorem 5.1.2 with $\tau_1 = 1$ and $\tau_2 = \tau$).

**Theorem 5.2.1.** *With 2 candidates in a metric space, if we know voters' preferences and a strength threshold* $\tau$*, Weighted Majority Rule 1 has a distortion of at most* $\delta = \max\{\frac{\tau+2}{\tau}, \frac{3\tau-1}{\tau+1}\}$.

*Proof.* Denote the set of voters prefer $P$ with preference strengths $> \tau$ as $A_2$, and with preference strengths $\leq \tau$ as $A_1$. Also denote the set of voters prefer $Q$ with preference strengths $> \tau$ as $B_2$, and with preference strengths $\leq \tau$ as $B_1$. Without loss of generality, suppose we choose $P$ as the winner by our weighted majority rule. It means that if $\tau \geq \sqrt{2}+1$, $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \geq |B_1| + \frac{\tau+1}{\tau-1}|B_2|$, and for $\tau < \sqrt{2}+1$, $\tau|A_2| + |A_1| \geq |B_1| + \tau|B_2|$.
Proof Sketch and Main Idea: For all voters, consider their individual ratio of $\frac{d(i,P)}{d(i,Q)}$, regardless of which candidate they prefer. For voters who prefer $P$ this is their preference strength, and for voters who prefer $Q$ this is the reciprocal of their preference strength. If for all voters this was less than $\delta$, then clearly we have a distortion of at most $\delta$ by just summing them up. However, for some voters this ratio is higher and for others it is lower. If we think of charging $SC(P)$ to $SC(Q)$, we should charge the voters for whom this ratio is lower to the voters for whom this ratio is higher. Clearly, for any voters who prefer $P$ this ratio is less than 1 and so it is less than $\delta$. For voters who prefer $Q$, some voters with weak preferences will allow us to save charge while others with stronger preferences will use up the extra charge. However, charging the voters to other voters seems quite difficult in this setting. The main new technique in our proof is to use $d(P, Q)$ as a sort of numeraire or store of value. We first perform the charging for all voters for whom this ratio is small, and we use $d(P, Q)$ to quantify how much extra charge is saved. We then show that this quantity of charge stored in terms of $d(P, Q)$ is sufficient to expend the charge from the remaining voters, yielding a

distortion at most $\delta$.

We first show some lemmas to bound $d(i,P)$ by $d(i,Q)$ and $d(P,Q)$ for every voter $i$.

**Lemma 5.2.2.** *$\forall i \in A_2$, for any $\delta \geq 1$, $d(i,P) \leq \delta d(i,Q) - \frac{\delta\tau-1}{\tau+1}d(P,Q)$.*

*Proof.* $\forall i \in A_2$, $d(i,P) \leq \frac{1}{\tau}d(i,Q)$. By the triangle inequality,

$$d(P,Q) \leq d(i,P) + d(i,Q) \leq \frac{1}{\tau}d(i,Q) + d(i,Q) = \frac{1+\tau}{\tau}d(i,Q)$$

Thus $d(i,Q) \geq \frac{\tau}{\tau+1}d(P,Q)$. $\forall i \in A_2$,

$$\begin{aligned} d(i,P) &\leq \frac{1}{\tau}d(i,Q) \\ &= \delta d(i,Q) - (\delta - \frac{1}{\tau})d(i,Q) \\ &\leq \delta d(i,Q) - \frac{\delta\tau - 1}{\tau+1}d(P,Q) \end{aligned}$$

□

**Lemma 5.2.3.** *$\forall i \in A_1$, for any $\delta \geq 1$, $d(i,P) \leq \delta d(i,Q) - \frac{\delta-1}{2}d(P,Q)$.*

*Proof.* $\forall i \in A_1$, by the triangle inequality,

$$d(P,Q) \leq d(i,P) + d(i,Q) \leq d(i,Q) + d(i,Q) = 2d(i,Q)$$

Thus $d(i,Q) \geq \frac{1}{2}d(P,Q)$. $\forall i \in A_1$,

$$\begin{aligned} d(i,P) &\leq d(i,Q) \\ &= \delta d(i,Q) - (\delta - 1)d(i,Q) \\ &\leq \delta d(i,Q) - \frac{\delta - 1}{2}d(P,Q) \end{aligned}$$

□

**Lemma 5.2.4.** *$\forall j \in B_1$, for any $1 \leq \delta \leq \tau$, $d(j,P) \leq \delta d(j,Q) + \frac{\tau-\delta}{\tau-1}d(P,Q)$.*
*$\forall j \in B_1$, for any $\delta > \tau$, $d(j,P) \leq \delta d(j,Q) - \frac{\delta-\tau}{\tau+1}d(P,Q)$.*

*Proof.* First consider the case that $1 \leq \delta \leq \tau$.

$\forall j \in B_1$, $d(j,P) \leq \tau d(j,Q)$. Also, by the triangle inequality, $d(j,P) \leq d(j,Q) + d(P,Q)$. By a linear combination of these two inequalities,

$$\begin{aligned} d(j,P) &\leq \frac{\delta-1}{\tau-1}\tau d(j,Q) + (1 - \frac{\delta-1}{\tau-1})(d(j,Q) + d(P,Q)) \\ &\leq \delta d(j,Q) + \frac{\tau-\delta}{\tau-1} d(P,Q) \end{aligned}$$

Then consider the case that $\delta > \tau$.

$\forall j \in B_1$, $d(j,P) \leq \tau d(j,Q)$. By the triangle inequality,

$$d(P,Q) \leq d(j,P) + d(j,Q) \leq \tau d(j,Q) + d(j,Q) = (1+\tau)d(j,Q)$$

Thus $d(j,Q) \geq \frac{1}{1+\tau} d(P,Q)$. $\forall j \in B_1$,

$$\begin{aligned} d(j,P) &\leq \tau d(j,Q) \\ &= \delta d(j,Q) - (\delta - \tau) d(j,Q) \\ &\leq \delta d(j,Q) - \frac{\delta-\tau}{\tau+1} d(P,Q) \end{aligned}$$

□

**Lemma 5.2.5.** $\forall j \in B_2$, $d(j,P) \leq d(j,Q) + d(P,Q)$.

*Proof.* This lemma follows directly by the triangle inequality. □

Using the four lemmas above, sum up for all voters, for any $\delta > \tau$,

$$\begin{aligned}
&\sum_{i\in A_1} d(i,P) + \sum_{i\in A_2} d(i,P) + \sum_{j\in B_1} d(j,P) + \sum_{j\in B_2} d(j,P) \\
&\le \delta \sum_{i\in A_1} d(i,Q) - |A_1|\frac{\delta-1}{2} d(P,Q) + \delta \sum_{i\in A_2} d(i,Q) - |A_2|\frac{\delta\tau-1}{\tau+1} d(P,Q) \\
&+ \delta \sum_{j\in B_1} d(j,Q) - |B_1|\frac{\delta-\tau}{\tau+1} d(P,Q) + \sum_{j\in B_2} d(j,Q) + |B_2| d(P,Q) \\
&\le \delta \sum_{i} d(i,Q) + (-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} - |B_1|\frac{\delta-\tau}{\tau+1} + |B_2|) d(P,Q)
\end{aligned} \tag{5.3}$$

Similarly, for any $1 \le \delta \le \tau$,

$$\sum_i d(i,P) \le \delta \sum_i d(i,Q) + (-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2|) d(P,Q) \tag{5.4}$$

Now we prove Theorem 5.2.1 by considering two cases: $\tau \ge \sqrt{2}+1$ and $\tau < \sqrt{2}+1$.

**Case 1,** $\tau \ge \sqrt{2}+1$**, and** $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \ge |B_1| + \frac{\tau+1}{\tau-1}|B_2|$

We prove the distortion is at most $\frac{3\tau-1}{\tau+1}$ in this case. Set $\delta = \frac{3\tau-1}{\tau+1}$. Note that when $\tau \ge 1$, $\delta = \frac{3\tau-1}{\tau+1} \le \tau$. By inequality 5.4, if we can prove $(-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2|) \le 0$, then $\sum_i d(i,P) \le \delta \sum_i d(i,Q)$.

When $\delta = \frac{3\tau-1}{\tau+1}$,

$$\begin{aligned}
&-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2| \\
&= -\frac{\tau-1}{\tau+1}|A_1| - \frac{3\tau^2-2\tau-1}{(\tau+1)^2}|A_2| + \frac{\tau-1}{\tau+1}|B_1| + |B_2| \\
&\le -\frac{\tau-1}{\tau+1}|A_1| - |A_2| + \frac{\tau-1}{\tau+1}|B_1| + |B_2| \\
&\le 0
\end{aligned}$$

The second to last line follows because $\frac{3\tau^2-2\tau-1}{(\tau+1)^2} \ge 1$ when $\tau \ge \sqrt{2}+1$. The last line follows because $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \ge |B_1| + \frac{\tau+1}{\tau-1}|B_2|$.

**Case 2,** $\tau < \sqrt{2}+1$**, and** $\tau|A_2| + |A_1| \ge |B_1| + \tau|B_2|$

We prove the distortion is at most $\frac{\tau+2}{\tau}$ in this case. Set $\delta = \frac{\tau+2}{\tau}$. Furthermore, we consider two subcases that $1 \le \tau < 2$ and $2 \le \tau < \sqrt{2}+1$.

**Case 2.1,** $2 \le \tau < \sqrt{2}+1$

When $2 \le \tau < \sqrt{2}+1$ and $\delta = \frac{\tau+2}{\tau}$, it is easy to show that $1 \le \delta \le \tau$. By inequality 5.4, if we can prove $(-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2|) \le 0$, then $\sum_i d(i,P) \le \delta \sum_i d(i,Q)$.

When $\delta = \frac{\tau+2}{\tau}$,

$$\begin{aligned}
& -|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2| \\
&= -\frac{1}{\tau}|A_1| - |A_2| + \frac{\tau^2-\tau-2}{\tau(\tau-1)}|B_1| + |B_2| \\
&\le -\frac{1}{\tau}|A_1| - |A_2| + \frac{1}{\tau}|B_1| + |B_2| \\
&\le 0
\end{aligned}$$

The second to last line follows because $\frac{\tau^2-\tau-2}{\tau(\tau-1)} \le \frac{1}{\tau}$ when $2 \le \tau < \sqrt{2}+1$. The last line follows because $\tau|A_2| + |A_1| \ge |B_1| + \tau|B_2|$.

**Case 2.2,** $1 \le \tau < 2$

Because $1 \le \tau < 2$ and $\delta = \frac{\tau+2}{\tau}$, it is easy to show that $\delta > \tau$. By inequality 5.3, if we can prove $(-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} - |B_1|\frac{\delta-\tau}{\tau+1} + |B_2|) \le 0$, then $\sum_i d(i,P) \le \delta \sum_i d(i,Q)$. When $\delta = \frac{\tau+2}{\tau}$,

$$\begin{aligned}
& -|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} - |B_1|\frac{\delta-\tau}{\tau+1} + |B_2| \\
&= -\frac{1}{\tau}|A_1| - |A_2| + \frac{\tau-2}{\tau}|B_1| + |B_2| \\
&\le -\frac{1}{\tau}|A_1| - |A_2| + \frac{1}{\tau}|B_1| + |B_2| \\
&\le 0
\end{aligned}$$

The second to last line follows because $\frac{\tau-2}{\tau} < 0 < \frac{1}{\tau}$ when $1 \le \tau < 2$. The last line follows because $\tau|A_2| + |A_1| \ge |B_1| + \tau|B_2|$.

Thus, we have shown that the distortion is at most $\frac{3\tau-1}{\tau+1}$ when $\tau \ge \sqrt{2}+1$, and at most $\frac{\tau+2}{\tau}$ when $\tau < \sqrt{2}+1$. Note that $\frac{3\tau-1}{\tau+1} \ge \frac{\tau+2}{\tau}$ when $\tau \ge \sqrt{2}+1$, and $\frac{3\tau-1}{\tau+1} < \frac{\tau+2}{\tau}$ when $\tau < \sqrt{2}+1$. Thus, the distortion of the weighted majority rule in this setting is $\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\}$. □

Note that Weighted Majority Rule 1 is not the only rule that gives the optimal distortion for two candidates. Consider the following simpler rule:

**Weighted Majority Rule 2.** *Given voters' preferences and a threshold* $\tau$ *for two candidates, assign weight* $\frac{\tau+1}{\tau-1}$ *to all the voters with preference strengths* $> \tau$ *and weight* $1$ *to all the voters with preference strengths* $\leq \tau$.

This rule gives the same distortion as Weighted Majority Rule 1 for two candidates, as we prove below. When extending these rules to more than 2 candidates, however, Weighted Majority Rule 1 allows us to form better mechanisms, thus sacrificing a small amount of simplicity for an improvement in distortion. We discuss this in the next section.

**Theorem 5.2.6.** *Weighted Majority Rule 2 has a distortion of at most* $\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\}$.

*Proof.* Denote the set of voters prefer $P$ with preference strengths $> \tau$ as $A_2$, and with preference strengths $\leq \tau$ as $A_1$. Also denote the set of voters prefer $Q$ with preference strengths $> \tau$ as $B_2$, and with preference strengths $\leq \tau$ as $B_1$. Without loss of generality, suppose we choose $P$ as the winner by Weighted Majority Rule 2. Thus, $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \geq |B_1| + \frac{\tau+1}{\tau-1}|B_2|$.

Similar to the proof of Theorem 5.2.1, we discuss three cases based on different values of $\tau$.

**Case 1,** $1 \leq \tau < 2$

Set $\delta = \frac{\tau+2}{\tau}$. Because $1 \leq \tau < 2$, it is easy to show that $\delta > \tau$. By inequality 5.3, if we can prove $(-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} - |B_1|\frac{\delta-\tau}{\tau+1} + |B_2|) \leq 0$, then $\sum_i d(i,P) \leq \delta \sum_i d(i,Q)$. When $\delta = \frac{\tau+2}{\tau}$,

$$
\begin{aligned}
&-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} - |B_1|\frac{\delta-\tau}{\tau+1} + |B_2| \\
&= -\frac{1}{\tau}|A_1| - |A_2| + \frac{\tau-2}{\tau}|B_1| + |B_2| \\
&\leq -\frac{\tau-1}{\tau+1}|A_1| - |A_2| + \frac{\tau-1}{\tau+1}|B_1| + |B_2| \\
&\leq 0
\end{aligned}
$$

The second to last line follows because $\frac{1}{\tau} \geq \frac{\tau-1}{\tau+1}$ and $\frac{\tau-2}{\tau} < 0 < \frac{\tau-1}{\tau+1}$ when $1 \leq \tau < 2$. The last line follows because $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \geq |B_1| + \frac{\tau+1}{\tau-1}|B_2|$.

**Case 2,** $2 \le \tau < \sqrt{2}+1$

Set $\delta = \frac{\tau+2}{\tau}$. When $2 \le \tau < \sqrt{2}+1$, it is easy to show that $1 \le \delta \le \tau$. By inequality 5.4, if we can prove $(-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2|) \le 0$, then $\sum_i d(i,P) \le \delta \sum_i d(i,Q)$.

When $\delta = \frac{\tau+2}{\tau}$,

$$
\begin{aligned}
& -|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2| \quad = -\frac{1}{\tau}|A_1| - |A_2| + \frac{\tau^2-\tau-2}{\tau(\tau-1)}|B_1| + |B_2| \\
& \le -\frac{\tau-1}{\tau+1}|A_1| - |A_2| + \frac{\tau-1}{\tau+1}|B_1| + |B_2| \\
& \le 0
\end{aligned}
$$

The second to last line follows because $\frac{1}{\tau} \ge \frac{\tau-1}{\tau+1}$ and $\frac{\tau^2-\tau-2}{\tau(\tau-1)} \le \frac{\tau-1}{\tau+1}$ when $2 \le \tau < \sqrt{2}+1$. The last line follows because $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \ge |B_1| + \frac{\tau+1}{\tau-1}|B_2|$.

**Case 3,** $\tau \ge \sqrt{2}+1$

Set $\delta = \frac{3\tau-1}{\tau+1}$. Note that when $\tau \ge \sqrt{2}+1$, $1 \le \delta \le \tau$. By inequality 5.4, if we can prove $(-|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2|) \le 0$, then $\sum_i d(i,P) \le \delta \sum_i d(i,Q)$.

When $\delta = \frac{3\tau-1}{\tau+1}$,

$$
\begin{aligned}
& -|A_1|\frac{\delta-1}{2} - |A_2|\frac{\delta\tau-1}{\tau+1} + |B_1|\frac{\tau-\delta}{\tau-1} + |B_2| \\
& = -\frac{\tau-1}{\tau+1}|A_1| - \frac{3\tau^2-2\tau-1}{(\tau+1)^2}|A_2| + \frac{\tau-1}{\tau+1}|B_1| + |B_2| \\
& \le -\frac{\tau-1}{\tau+1}|A_1| - |A_2| + \frac{\tau-1}{\tau+1}|B_1| + |B_2| \\
& \le 0
\end{aligned}
$$

The second to last line follows because $\frac{3\tau^2-2\tau-1}{(\tau+1)^2} \ge 1$ when $\tau \ge \sqrt{2}+1$. The last line follows because $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \ge |B_1| + \frac{\tau+1}{\tau-1}|B_2|$.

Thus, we proved that the distortion is at most $\frac{\tau+2}{\tau}$ when $1 \le \tau < \sqrt{2}+1$, and at most $\frac{3\tau-1}{\tau+1}$ when $\tau \ge \sqrt{2}+1$. □

### 5.2.2 Multiple Candidates (Given Preferences and a Threshold $\tau$)

In this section, we discuss mechanisms with small distortion for multiple ($\ge 3$) candidates. We assume that we are given the ordinal preference ordering of each voter for all candidates, as well as an indication whether, for every pair of candidates, the voter has a strong preference ($> \tau$), or a weak preference ($\leq \tau$). While this certainly requires more than a single bit of information for every voter, we believe that such data is reasonably possible to collect: it is usually easy for voters to express whether they prefer option A to option B *strongly* or *weakly*, as opposed to trying to quantify exactly how strong their preference is. In reality we would need to compare only the obviously front-runner candidates in this way, and would not actually need this thresholded knowledge for *every* pair of candidates. As discussed in Section 1, this information could also be reasonably estimated from other sources, such as the amount of monetary donations, attendance to political rallies, the amount of "buzz" on social media, etc.

The mechanisms we consider are as follows. First, we create a weighted majority graph by choosing pairwise winners using Majority Rule 1. Then we study the distortion of the winner(s) in the uncovered set [92] in this majority graph. Recall that if a candidate $P$ is in the uncovered set, it means that for any candidate $Z$, either $P$ beats $Z$ directly, or there exists another candidate $Q$ such that $P$ beats $Q$, and $Q$ beats $Z$. The uncovered set is always known to be non-empty, and for example the Copeland mechanism always chooses a candidate in the uncovered set.

We begin with the following useful lemma due to Goel at al. [14]

**Lemma 5.2.7.** (Goel et al, 2017)
*If a majority of voters prefer $P$ to $Q$, then $SC(P) \leq 2 \cdot SC(Z) + SC(Q)$ for any other possible candidate $Z$.*

We first show that while this lemma certainly does not hold for all pairwise majority rules, this lemma can be generalized specifically for Majority Rule 1. We then use this to prove bounds on the distortion of the above "uncovered set" mechanisms. This lemma is precisely why we use Majority Rule 1 instead of, for example, simpler conditions such as Majority Rule 2, since while their distortion for two candidates remains the same, the theorem below fails to hold.

**Theorem 5.2.8.** *If Majority Rule 1 selects $P$ over $Q$, then $SC(P) \leq 2 \cdot SC(Z) + SC(Q)$ where $Z$ can be any point in the metric space.*

*Proof.* We use the same notation as before. Let $A_1$ denote a subset of voters that prefer $P$ to $Q$ with preference strengths $\leq \tau$, and let $A_2$ denote a subset of voters that prefer $P$ to $Q$ with preference strengths $> \tau$. Also $B_1$ denote a subset of voters prefer $Q$ to $P$ with preference strengths $\leq \tau$, and let $B_2$ denote a subset of voters prefer $Q$ to $P$ with preference strengths $> \tau$. Without loss of generality, suppose we choose $P$ as the winner by our weighted majority rule. It means that if $\tau \geq \sqrt{2}+1$, $\frac{\tau+1}{\tau-1}|A_2| + |A_1| \geq |B_1| + \frac{\tau+1}{\tau-1}|B_2|$, and if $\tau < \sqrt{2}+1$, $\tau|A_2| + |A_1| \geq |B_1| + \tau|B_2|$.

From Lemma 5.2.7, we know that if $|A_2|+|A_1| \geq |B_1|+|B_2|$, then $SC(P) \leq 2\cdot SC(Z) + SC(Q)$. Consider the case that $|A_2| + |A_1| < |B_1| + |B_2|$, it is not possible that $|A_2| < |B_2|$ and $|A_1| >= |B_1|$, because $A_2$ and $B_2$ have heavier weight than $A_1$ and $B_1$. Thus the only case left is $|A_2| >= |B_2|$ and $|A_1| < |B_1|$.

We separate the voters in $A_2$ into two subsets $A_2'$ and $A_2 - A_2'$, such that $|A_2'| = |B_2|$. Similarly, we separate the voters in $B_1$ into two subsets $B_1'$ and $B_1 - B_1'$, such that $|B_1'| = |A_1|$.

**Case 1.** $\tau \geq \sqrt{2}+1$

By our weighted majority rule,

$$\frac{\tau+1}{\tau-1}|A_2| + |A_1| \geq |B_1| + \frac{\tau+1}{\tau-1}|B_2|$$
$$\frac{\tau+1}{\tau-1}(|A_2'| + |A_2 - A_2'|) + |B_1| \geq (|B_1'| + |B_1 - B_1'|) + \frac{\tau+1}{\tau-1}|B_2|$$
$$\frac{\tau+1}{\tau-1}|A_2 - A_2'| \geq |B_1 - B_1'|$$

First, bound $\sum\limits_{i \in B_1 - B_1'} d(i,P)$, by the triangle inequality,

$$\begin{aligned}
\sum_{i \in B_1 - B_1'} d(i,P) &\leq \sum_{B_1 - B_1'} d(i,Z) + \sum_{B_1 - B_1'} d(P,Z) \\
&= \sum_{B_1 - B_1'} d(i,Z) + |B_1 - B_1'|\, d(P,Z) \\
&\leq \sum_{B_1 - B_1'} d(i,Z) + \frac{\tau+1}{\tau-1}|A_2 - A_2'|\, d(P,Z) \\
&= \sum_{B_1 - B_1'} d(i,Z) + \frac{\tau+1}{\tau-1} \sum_{A_2 - A_2'} d(P,Z)
\end{aligned}$$

The second to last line follows because $\frac{\tau+1}{\tau-1}|A_2 - A'_2| \geq |B_1 - B'_1|$. Using the triangle inequality again, $\forall i \in A_2 - A'_2$, $d(P,Z) \leq d(i,P) + d(i,Z)$, and also note that $d(i,P) \leq \frac{1}{\tau}d(i,Q)$. Thus,

$$\begin{aligned}
\sum_{i \in B_1 - B'_1} d(i,P) &\leq \sum_{B_1 - B'_1} d(i,Z) + \frac{\tau+1}{\tau-1} \sum_{A_2 - A'_2} d(P,Z) \\
&\leq \sum_{B_1 - B'_1} d(i,Z) + \frac{\tau+1}{\tau-1} \sum_{A_2 - A'_2} (d(i,P) + d(i,Z)) \\
&\leq \sum_{B_1 - B'_1} d(i,Z) + \frac{\tau+1}{\tau-1} \sum_{A_2 - A'_2} (\frac{1}{\tau}d(i,Q) + d(i,Z)) \\
&= \frac{\tau+1}{\tau-1} \sum_{A_2 - A'_1} d(i,Z) + \sum_{B_1 - B'_1} d(i,Z) + \frac{\tau+1}{\tau(\tau-1)} \sum_{A_2 - A'_2} d(i,Q)
\end{aligned}$$

Multiply both sides by $\frac{\tau-1}{\tau}$,

$$\frac{\tau-1}{\tau} \sum_{i \in B_1 - B'_1} d(i,P) \leq \frac{\tau+1}{\tau} \sum_{A_2 - A'_1} d(i,Z) + \frac{\tau-1}{\tau} \sum_{B_1 - B'_1} d(i,Z) + \frac{\tau+1}{\tau^2} \sum_{A_2 - A'_2} d(i,Q) \quad (5.5)$$

Also, $\forall i \in B_1 - B'_1$, $d(i,P) \leq \tau d(i,Q)$. So $\sum_{i \in B_1 - B'_1} d(i,P) \leq \tau \sum_{i \in B_1 - B'_1} d(i,Q)$. Divide both sides by $\tau$, we get $\frac{1}{\tau} \sum_{i \in B_1 - B'_1} d(i,P) \leq \sum_{i \in B_1 - B'_1} d(i,Q)$. Finally, $\forall i \in A_2 - A'_2$, bound $d(i,P)$ by $\frac{1}{\tau}d(i,Q)$. Together with Inequality 5.5,

$$\sum_{i\in A_2-A_2'} d(i,P) + \sum_{i\in B_1-B_1'} d(i,P)$$
$$\leq \frac{1}{\tau}\sum_{A_2-A_2'} d(i,Q) + \frac{1}{\tau}\sum_{i\in B_1-B_1'} d(i,P) + \frac{\tau-1}{\tau}\sum_{i\in B_1-B_1'} d(i,P)$$
$$\leq \frac{1}{\tau}\sum_{A_2-A_2'} d(i,Q) + \sum_{i\in B_1-B_1'} d(i,Q) + \frac{\tau+1}{\tau}\sum_{A_2-A_1'} d(i,Z) + \frac{\tau-1}{\tau}\sum_{B_1-B_1'} d(i,Z)$$
$$+\frac{\tau+1}{\tau^2}\sum_{A_2-A_2'} d(i,Q)$$
$$= \frac{\tau+1}{\tau}\sum_{A_2-A_1'} d(i,Z) + \frac{\tau-1}{\tau}\sum_{B_1-B_1'} d(i,Z) + \frac{2\tau+1}{\tau^2}\sum_{A_2-A_2'} d(i,Q) + \sum_{i\in B_1-B_1'} d(i,Q)$$
$$\leq 2(\sum_{i\in A_2-A_2'} d(i,Z) + \sum_{i\in B_1-B_1'} d(i,Z)) + \sum_{i\in A_2-A_2'} (i,Q) + \sum_{i\in B_1-B_1'} d(i,Q)$$

The last line follows because $\frac{2\tau+1}{\tau^2} \leq 1$ when $\tau \geq \sqrt{2}+1$.

**Case 2.** $\tau < \sqrt{2}+1$

By our weighted majority rule,

$$\tau|A_2| + |A_1| \geq |B_1| + \tau|B_2|$$
$$\tau(|A_2'| + |A_2 - A_2'|) + |A_1| \geq (|B_1'| + |B_1 - B_1'|) + \tau|B_2|$$
$$\tau|A_2 - A_2'| \geq |B_1 - B_1'|$$

The proof is almost the same as **Case 1**, except that we use the inequality above to bound the ratio between $|B_1 - B_1'|$ and $|A_2 - A_2'|$. Similar to Inequality 5.5, we get:

$$\frac{\tau-1}{\tau}\sum_{i\in B_1-B_1'} d(i,P) \leq (\tau-1)\sum_{A_2-A_1'} d(i,Z) + \frac{\tau-1}{\tau}\sum_{B_1-B_1'} d(i,Z) + \frac{\tau-1}{\tau}\sum_{A_2-A_2'} d(i,Q) \quad (5.6)$$

Then bound $\sum_{i\in A_2-A_2'} d(i,P) + \sum_{i\in B_1-B_1'} d(i,P)$ similarly to **Case 1**,

$$\sum_{i\in A_2-A_2'} d(i,P) + \sum_{i\in B_1-B_1'} d(i,P)$$
$$= \frac{1}{\tau}\sum_{A_2-A_2'} d(i,Q) + \frac{1}{\tau}\sum_{i\in B_1-B_1'} d(i,P) + \frac{\tau-1}{\tau}\sum_{i\in B_1-B_1'} d(i,P)$$
$$\leq \frac{1}{\tau}\sum_{A_2-A_2'} d(i,Q) + \sum_{i\in B_1-B_1'} d(i,Q) + (\tau-1)\sum_{A_2-A_1'} d(i,Z) + \frac{\tau-1}{\tau}\sum_{B_1-B_1'} d(i,Z)$$
$$+ \frac{\tau-1}{\tau}\sum_{A_2-A_2'} d(i,Q)$$
$$= (\tau-1)\sum_{A_2-A_1'} d(i,Z) + \frac{\tau-1}{\tau}\sum_{B_1-B_1'} d(i,Z) + \sum_{A_2-A_2'} d(i,Q) + \sum_{i\in B_1-B_1'} d(i,Q)$$
$$\leq 2(\sum_{i\in A_2-A_2'} d(i,Z) + \sum_{i\in B_1-B_1'} d(i,Z)) + \sum_{i\in A_2-A_2'} (i,Q) + \sum_{i\in B_1-B_1'} d(i,Q)$$

We have proved $\sum_{i\in A_2-A_2'+B_1-B_1'} (i,P) \leq \sum_{i\in A_2-A_2'+B_1-B_1'} (i,Q) + 2\sum_{i\in A_2-A_2'+B_1-B_1'} (i,Z)$ for any $\tau \geq 1$. And because $|A_2'| + |A_1| = |B_1'| + |B_2|$, by Lemma 5.2.7,

$$\sum_{i\in A_2'+A_1+B_1'+B_2} (i,P) \leq \sum_{i\in A_2'+A_1+B_1'+B_2} (i,Q) + 2\sum_{i\in A_2'+A_1+B_1'+B_2} (i,Z)$$

Putting everything together, $\sum_i (i,P) \leq \sum_i (i,Q) + 2\sum_i (i,Z)$. □

Now that we have the above theorem, it is easy to establish distortion bounds based on our weighted majority rule.

**Theorem 5.2.9.** *Suppose a weighted majority graph is formed by using Majority Rule 1 to choose pairwise winners. The distortion of the uncovered set of this graph is at most* $\min\{\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\}+2, \max\{(\frac{3\tau-1}{\tau+1})^2, (\frac{\tau+2}{\tau})^2\}\}$ *in the multiple candidates setting when given voters' ordinal preferences and a threshold* $\tau$.

*Proof.* Suppose the optimal candidate is $Z$. By definition, for any candidate $P$ in the uncovered set, either $P$ beats $Z$ directly or there exists a candidate $Q$, that $P$ beats $Q$ and $Q$ beats $Z$. And we know that the distortion between two candidates when one beats the other directly is at most $\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\}$, so it is straight forward that the distortion is at most $\max\{(\frac{3\tau-1}{\tau+1})^2, (\frac{\tau+2}{\tau})^2\}$ for any winner in the uncovered set.

Also, because $Q$ beats $Z$, so $SC(Q) \leq \max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\}SC(Z)$. By Theorem 5.2.8, we know that $SC(P) \leq (\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\} + 2)SC(Z)$. Thus, we can get a upper bound of distortion for the uncovered set of $\min\{\max\{\frac{3\tau-1}{\tau+1}, \frac{\tau+2}{\tau}\} + 2, \max\{(\frac{3\tau-1}{\tau+1})^2, (\frac{\tau+2}{\tau})^2\}\}$. □

### 5.2.3 Choosing the Best Threshold

What type of knowledge of the strengths in voter preferences is most useful and advantageous? If you could gather data about voter preferences in different ways, what should you aim for in order to reduce distortion? These are some of the questions which we wish to illuminate in this chapter.

Our results in the previous two sections shed some light on these decisions. First, it may be surprising (although it really shouldn't be) that knowing only information about very extreme voters (i.e., $\tau$ being high) or only about very indecisive voters ($\tau$ being very close to 1) does not help much when compared to only knowing the voters' ordinal preferences. Our results indicate, however, that the optimal thing to do is to differentiate between candidates with a lot of supporters who prefer them at least 2 times to other candidates (or more precisely, at least $1 + \sqrt{2}$ times), and candidates which have few such supporters. Our results indicate that by obtaining this information, we can improve the quality of the chosen candidate from a 3-approximation to only a 1.83 approximation (for 2 candidates), and from a 5-approximation to a 3.35-approximation (for three or more candidates). This is a huge improvement obtained with relatively little extra cost.

## 5.3 Undecided Voters: Working Without Knowing Voter Preferences

Suppose there are two candidates $P$ and $Q$ and for all voters with preference strength greater than threshold $\tau$, we know their preferred candidate. For all other voters we know *nothing* about their preferences. This is a strict generalization of the case where we just know voter preferences, since that is the case where $\tau = 1$. As with the case where we only know preferences, the only reasonable voting rule is to select the candidate preferred by more voters (in the case that there are only two candidates), out of those for whom we know preferences. This represents the case where voters abstain if their preference strength is not sufficiently high for them to be motivated enough to vote. In this section we consider mechanisms to deal with such undecided or unmotivated voters.

**Weighted Majority Rule 3.** *Given candidates $P$ and $Q$ and any single threshold $\tau \geq 1$, give all voters with preference strength at least $\tau$ a weight of 1 and all other voters a weight of 0. Select the candidate by weighted majority rule.*

**Theorem 5.3.1.** *With two candidates and only the preferences of voters with preference strength greater than $\tau$, Weighted Majority Rule 3 achieves a distortion of $\max\{\frac{\tau+2}{\tau}, \tau\}$, and no deterministic mechanism can do better.*

*Proof.* The proof is similar to that of Theorem 5.2.1, once again using $d(P,Q)$ as an intermediate value to charge possible voter distances to. Let $A$ be the set of voters who strongly prefer $P$. That is, $A = \{i : \frac{d(i,Q)}{d(i,P)} \geq \tau\}$. Similarly, define the set of voters who strongly prefer $Q$ as $B = \{j : \frac{d(j,P)}{d(j,Q)} \geq \tau\}$. Let the set of remaining voters, whose preference strengths are weaker than $\tau$ be denoted $C$. Without loss of generality, let $P$ be the winner over $Q$ because $|A| \geq |B|$.

**Lemma 5.3.2.** $\forall i \in A$, for any $\delta \geq 1$ *:* $d(i,P) \leq \delta d(i,Q) - \frac{\delta\tau-1}{\tau+1} d(P,Q)$.

*Proof.* $\forall i \in A$ we know that $d(i,P) \leq \frac{1}{\tau} d(i,Q)$.

It follows from triangle inequality that $d(P,Q) \leq d(i,P) + d(i,Q) \leq \frac{1}{\tau}(i,Q) + (i,Q) = \frac{\tau+1}{\tau}(i,Q)$.

For any $\delta \geq 1$ we when have

$$\begin{aligned} d(i,P) &\leq \frac{1}{\tau} d(i,Q) \\ &= \delta d(i,Q) - (\delta - \frac{1}{\tau}) d(i,Q) \\ &\leq \delta d(i,Q) - (\delta - \frac{1}{\tau})(\frac{\tau}{\tau+1}) d(P,Q) \\ &= \delta d(i,Q) - (\frac{\delta\tau - 1}{\tau+1}) d(P,Q) \end{aligned}$$

□

Recall that $d(j,P) \leq d(j,Q) + d(P,Q)$ from triangle inequality. It therefore follows from Lemma 5.3.2 that for any $\delta \geq 1$,

$$\sum_{i \in A} d(i,P) + \sum_{j \in B} (j,P) \leq \delta \sum_{i \in A} d(i,Q) - |A|(\frac{\delta\tau - 1}{\tau+1}) d(P,Q) + \sum_{j \in B} d(j,Q) + |B| d(P,Q)$$

Let $\delta = \max\{\frac{\tau+2}{\tau}, \tau\}$. We consider the two cases in which either of the two terms in this bound are the larger term.

**Case 1:** If $\tau \geq 2$ then $\delta = \tau$, and therefore

$$\begin{aligned}\sum_{i \in A} d(i,P) + \sum_{j \in B}(j,P) &\leq \delta \sum_{i \in A} d(i,Q) - |A|(\tau-1)d(P,Q) + \sum_{j \in B} d(j,Q) + |B|d(P,Q) \\ &\leq \delta \sum_{i \in A} d(i,Q) + \sum_{j \in B}(j,Q) \qquad \text{because} \quad |A| \geq |B|.\end{aligned}$$

**Case 2:** If $\tau < 2$ then $\delta = \frac{\tau+2}{\tau}$, and therefore

$$\begin{aligned}\sum_{i \in A} d(i,P) + \sum_{j \in B}(j,P) &\leq \delta \sum_{i \in A} d(i,Q) - |A|d(P,Q) + \sum_{j \in B} d(j,Q) + |B|d(P,Q) \\ &\leq \delta \sum_{i \in A} d(i,Q) + \sum_{j \in B}(j,Q) \qquad \text{because} \quad |A| \geq |B|.\end{aligned}$$

Lastly, we can see that this upper bound on $\delta$ is tight due to the lower bounds given by examples in Lemma 5.1.3 and Lemma 5.1.4. □

### 5.3.1 Choosing the Best Threshold

If we can only select a single threshold for voter preference strengths, which should we choose? Intuitively, this is analogous to determining how difficult it should be to vote. If it takes a little bit of effort to vote, then you know that the voters who actually do participate have a significant interest in the outcome. However, if the barriers to voting are too high, then the outcome can be decided by a small fraction of the voters and fails to capture their collective preferences as a whole (see Figure 5.1). In our setting the optimal choice of threshold is $\arg\min_{\tau}\{\max\{\frac{\tau+2}{\tau}, \tau\}\} = 2$, yielding a distortion of 2 (instead of 3 for the case when $\tau = 1$).

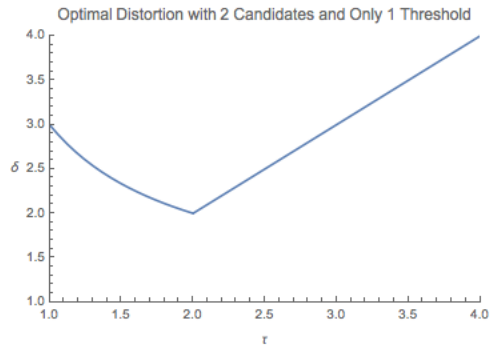


**Figure 5.1: Best achievable distortion for a single threshold $\tau$.**

### 5.3.2 Multiple Candidates (Given Only a Threshold $\tau$)

When there are more than two candidates, we study the distortion of the uncovered set.

**Theorem 5.3.3.** *With mutiple candidates and only the preferences of voters with preference strength greater than $\tau$, if Weighted Majority Rule 3 is used to choose pairwise winners, then the distortion of the uncovered set of this graph is at most* $\max\{(\frac{\tau+2}{\tau})^2, \tau^2\}$.

*Proof.* Suppose the optimal candidate is $Z$. By definition, for any candidate $P$ in the uncovered set, either $P$ beats $Z$ directly or there exists a candidate $Q$, such that $P$ beats $Q$ and $Q$ beats $Z$. And we know that the distortion between two candidates when one beats the other directly is at most $\max\{\frac{\tau+2}{\tau}, \tau\}$, so it is straight forward that the distortion is at most $\max\{(\frac{\tau+2}{\tau})^2, \tau^2\}$ for any winner in the uncovered set. □

Note that, unlike in Theorem 5.2.9, for this setting we have to settle for the trivial bound of squaring the distortion for $\geq 3$ candidates. This is because, unlike for the case with known preferences and a threshold, the property that $SC(P) \leq 2 \cdot SC(Z) + SC(Q)$ (Theorem 5.2.8) does not hold anymore. Consider the following example: there are three candidates $P$, $Q$, and $Z$, and there is only one voter $i$, that has a preference strength $< \tau$ between any pair of candidates, so we have no information whatsoever about this voter preferences. Without loss of generality, suppose we choose $P$ as the winner. The actual distances could be: $d(i,P) = \tau - \epsilon$, $d(i,Q) = 1$, and $d(i,Z) = 1$. As $\epsilon$ approaches 0, $SC(P) \approx \tau SC(Q)$, and also $SC(P) \approx \tau SC(Z)$. When $\tau$ is large, it is not possible to have $SC(P) \leq 2SC(Z) + SC(Q)$. Thus, we cannot bound $SC(P)$ in the multiple candidates setting by $SC(P) \leq 2 \cdot SC(Z) + SC(Q)$ as in Section 5.2.

## 5.4 Distortion with General Thresholds

In this section we generalize some of our results in the previous sections to deal with general preference strength thresholds. We are given thresholds $\{1 \leq \tau_1 < \tau_2 < \ldots < \tau_m\}$, and for every voter $i$ and pair of candidates $P$ and $Q$ we know the pair of thresholds between which the preference strength of $i$ falls into. In other words, the more thresholds we have, the less coarse our knowledge of voters preferences. We believe it is realistic to assume that we have one or two, perhaps three, such thresholds, and for most candidate pairs we can create a profile describing how devoted and fanatical their supporters are with respect to these thresholds. However, in this section we consider general sets of thresholds in order to provide bounds on distortion which are as general as possible. For convenience, we let $\tau_{m+1} = \infty$ and $\tau_0 = \frac{1}{\tau_1}$.

We begin as before, by analyzing the case with only 2 candidates $P$ and $Q$, and then extending our results to multiple candidates.

**Condition 5.4.1.** *Let* $\delta = \max\limits_{0 \leq l \leq m} \{\frac{\tau_l \tau_{l+1} + 2\tau_{l+1} - 1}{\tau_l \tau_{l+1} + 1}\}$*. Find* $k$ *such that* $\tau_k \leq \delta < \tau_{k+1}$*.* $P$ *wins only if* $\sum\limits_{l=k}^{m} (\frac{\tau_{l+1} - \delta}{\tau_{l+1} - 1}) |B_l| \leq \sum\limits_{l=1}^{m} (\frac{\delta \tau_l - 1}{\tau_l + 1}) |A_l| + \sum\limits_{l=1}^{k-1} |B_l| (\frac{\delta - \tau_{l+1}}{\tau_{l+1} + 1})$ *and* $Q$ *wins only if* $\sum\limits_{l=k}^{m} (\frac{\tau_{l+1} - \delta}{\tau_{l+1} - 1}) |A_l| \leq \sum\limits_{l=1}^{m} (\frac{\delta \tau_l - 1}{\tau_l + 1}) |B_l| + \sum\limits_{l=1}^{k-1} |A_l| (\frac{\delta - \tau_{l+1}}{\tau_{l+1} + 1})$*.*

The above is not a specific voting rule, but is instead a set of voting rules. We prove below that any rule obeying the above condition has distortion at most $\delta$, and that we can always form a rule satisfying this condition. Note that such a value of $0 < k \leq m$ always exists because distortion is at least $\tau_1$ (since taking the term for $l = 0$ gives $\tau_1$). It may be that $k = m$, where $\tau_m \leq \delta$.

**Theorem 5.4.2.** *Any single-winner voting rule over two candidates which satisfies Condition 5.4.1 has distortion* $\delta = \max\limits_{0 \leq l \leq m} \{\frac{\tau_l \tau_{l+1} + 2\tau_{l+1} - 1}{\tau_l \tau_{l+1} + 1}\}$ *and no deterministic mechanism can do better.*

*Proof.*

*Outline:* First, we prove the upper bound on distortion. We want to show that if $P$ wins then $SC(P) = \sum\limits_{l=1}^{m} \sum\limits_{i \in A_l} d(i, P) + \sum\limits_{l=1}^{m} \sum\limits_{j \in B_l} d(j, P) + \sum\limits_{k \in C} d(k, P) \leq \delta \big( \sum\limits_{l=1}^{m} \sum\limits_{i \in A_l} d(i, Q) + \sum\limits_{l=1}^{m} \sum\limits_{j \in B_l} d(j, Q) + \sum\limits_{k \in C} d(j, Q) \big) = \delta SC(Q)$. We prove this by using four lemmas which each establish an upper bound on the social cost accrued to $P$ by a subset of the voters. To do this we use $d(P, Q)$ as a sort of numeraire or store of value. Summing over the three inequalities in these lemmas proves the upper bound on distortion as long as Condition 5.4.1 is met. Tightness follows from Theorem 5.1.2 in Section 5.1.1.

**Lemma 5.4.3.** *If $P$ wins then* $\sum_{k\in C} d(k,P) \leq \sum_{k\in C} \delta d(k,Q)$

*Proof.*
$\forall k \in C : d(k,P) \leq \tau_1 \, d(k,Q)$ and we know $\tau_1 \leq \delta$ by our choice of $\delta$. □

**Lemma 5.4.4.** *If $P$ wins then* $\sum_{l=1}^{m}\sum_{i\in A_l} d(i,P) \leq \sum_{l=1}^{m}\sum_{i\in A_l} \delta d(i,Q) - \sum_{l=1}^{m}\sum_{i\in A_l} (\frac{\delta\tau_l-1}{\tau_l+1}) d(P,Q)$

*Proof.* Recall from the definition of $A_l$ that $\forall l \leq m, \forall i \in A_l : d(i,P) \leq \frac{1}{\tau_l} d(i,Q)$.

This implies $\forall l \leq m, \forall i \in A_l : d(P,Q) \leq d(i,P) + d(i,Q) \leq \frac{\tau_l+1}{\tau_l} d(i,Q)$.

It follows that

$$
\begin{aligned}
\sum_{l=1}^{m}\sum_{i\in A_l} d(i,P) &\leq \sum_{l=1}^{m}\sum_{i\in A_l} \frac{1}{\tau_l} d(i,Q) \\
&= \sum_{l=1}^{m}\sum_{i\in A_l} \Big(\delta d(i,Q) - (\delta - \frac{1}{\tau_l}) d(i,Q)\Big) \\
&\leq \sum_{l=1}^{m}\sum_{i\in A_l} \Big(\delta d(i,Q) - (\frac{\delta\tau_l - 1}{\tau_l})(\frac{\tau_l}{\tau_l+1}) d(P,Q)\Big) \\
&= \sum_{l=1}^{m}\sum_{i\in A_l} \delta d(i,Q) - \sum_{l=1}^{m}\sum_{i\in A_l} \Big(\frac{\delta\tau_l - 1}{\tau_l+1}\Big) d(P,Q)
\end{aligned}
$$

□

**Lemma 5.4.5.** *If $P$ wins then* $\sum_{l=1}^{k-1}\sum_{j\in B_l} d(j,P) \leq \sum_{l=1}^{k-1}\sum_{j\in B_l} \delta d(j,Q) - \sum_{l=1}^{k-1}\sum_{j\in B_l} (\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1}) d(P,Q)$

*Proof.* Recall from the definition of $B_l$ that $\forall l < k, \forall j \in B_l : d(j,P) \leq \tau_{l+1} d(j,Q)$.

This implies $\forall l < k, \forall j \in B_l : d(P,Q) \leq d(j,P) + d(j,Q) \leq (\tau_{l+1}+1) d(j,Q)$.

It follows that

$$\begin{aligned}
\sum_{l=1}^{k-1}\sum_{j\in B_l} d(j,P) &\le \sum_{l=1}^{k}\sum_{j\in B_l} \tau_{l+1} d(j,Q) \\
&= \sum_{l=1}^{k-1}\sum_{j\in B_l} \Big(\tau_{l+1} d(j,Q) + (\delta-\tau_{l+1})d(j,Q) - (\delta-\tau_{l+1})d(j,Q)\Big) \\
&= \sum_{l=1}^{k-1}\sum_{j\in B_l} \delta d(j,Q) - \sum_{l=1}^{k}\sum_{j\in B_l}(\delta-\tau_{l+1})d(j,Q) \\
&\le \sum_{l=1}^{k-1}\sum_{j\in B_l} \delta d(j,Q) - \sum_{l=1}^{k}\sum_{j\in B_l}\Big(\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1}\Big)d(P,Q)
\end{aligned}$$

□

**Lemma 5.4.6.** *If $P$ wins then* $\sum_{l=k}^{m}\sum_{j\in B_l} d(j,P) \le \sum_{l=k}^{m}\sum_{j\in B_l}\delta d(j,Q) + \sum_{l=k}^{m}\sum_{j\in B_l}(\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})d(P,Q)$

*Proof.* Recall from the definition of $B_l$ that $\forall l \ge k, \forall j \in B_l : d(j,P) \le \tau_{l+1} d(j,Q)$.

From triangle inequality $\forall j : d(j,P) \le d(j,Q) + d(P,Q)$.

Together these imply, $\forall l \ge k, \forall j \in B_l : d(j,P) \le x\tau_{l+1}d(j,Q) + (1-x)\big(d(j,Q) + d(P,Q)\big)$ for any $0 \le x \le 1$.

Below, for each $l \ge k$ we choose $x = \frac{\delta-1}{\tau_{l+1}-1} \le 1$.

It follows that

$$\begin{aligned}
\sum_{l=k}^{m}\sum_{j\in B_l} d(j,P) &\le \sum_{l=k}^{m}\sum_{j\in B_l} (\frac{\delta-1}{\tau_{l+1}-1})\tau_{l+1}d(j,Q) + \big(1-\frac{\delta-1}{\tau_{l+1}-1}\big)\big(d(j,Q)+d(P,Q)\big) \\
&= \sum_{l=k}^{m}\sum_{j\in B_l} \delta d(j,Q) + \sum_{l=k}^{m}\sum_{j\in B_l}\big(1-\frac{\delta-1}{\tau_{l+1}-1}\big)d(P,Q) \\
&= \sum_{l=k}^{m}\sum_{j\in B_l} \delta d(j,Q) + \sum_{l=k}^{m}\sum_{j\in B_l}(\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})d(P,Q)
\end{aligned}$$

□

By summing over the inequalities in the four preceding lemmas, we have

$$SC(P) \le \delta SC(Q) + d(P,Q)\Big(\sum_{l=k}^{m}\sum_{j\in B_l}(\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1}) - \sum_{l=1}^{k}\sum_{j\in B_l}\Big(\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1}\Big) - \sum_{l=1}^{m}\sum_{i\in A_l}\Big(\frac{\delta\tau_l-1}{\tau_l+1}\Big)\Big)$$

If Condition 5.4.1 holds when $P$ wins the $d(P,Q)$ term on the RHS is non-positive, and we have $SC(P) \leq \delta SC(Q)$ as desired. □

We have now shown that any voting rule obeying the above condition has distortion at most $\delta$. We now prove that for any instance, selecting one of the two candidates *must* satisfy Condition 5.4.1, so we can construct resolute single-winner voting rules which satisfy this condition. Last, we provide a specific weighted majority rule which always satisfies Condition 5.4.1.

**Lemma 5.4.7.** *Given any instance, i.e., a set of voters, two candidates, and a set of thresholds, selecting at least one of the candidates must satisfy Condition 5.4.1.*

*Proof.* Put another way, at least one of the two inequalities in Condition 5.4.1 must hold, so there can be no instance in which neither candidate can be selected.

Suppose $\delta \geq \max_{0 \leq l \leq m} \{\frac{\tau_l \tau_{l+1} + 2\tau_{l+1} - 1}{\tau_l \tau_{l+1} + 1}\}$.

By moving over the denominator, this can be rewritten as

$$\forall l \leq m : \delta(\tau_l \tau_{l+1} + 1) \geq \tau_l \tau_{l+1} + 2\tau_{l+1} - 1$$

or

$$\forall l \leq m : (\delta\tau_l - 1)(\tau_{l+1} - 1) - (\tau_l + 1)(\tau_{l+1} - \delta) \geq 0.$$

We can divide both sizes to obtain

$$\forall l \leq m : \frac{(\delta\tau_l - 1)(\tau_{l+1} - 1) - (\tau_l + 1)(\tau_{l+1} - \delta)}{(\tau_l + 1)(\tau_{l+1} - 1)} \geq 0$$

and simplify to get,

$$\forall l \leq m : \frac{\delta\tau_l - 1}{\tau_l + 1} - \frac{\tau_{l+1} - \delta}{\tau_{l+1} - 1} \geq 0.$$

We can now express our inequality in terms of the sets of voters

$$\sum_{l=k}^{m} (|A_l| + |B_l|)(\frac{\delta\tau_l - 1}{\tau_l + 1} - \frac{\tau_{l+1} - \delta}{\tau_{l+1} - 1}) \geq 0$$

and separate to yield

$$\sum_{l=k}^{m}(|A_l|+|B_l|)(\frac{\delta\tau_l-1}{\tau_l+1}-\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})+\sum_{l=1}^{k-1}(|A_l|+|B_l|)(\frac{\delta\tau_l-1}{\tau_l+1}+\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1})\geq 0.$$

We can separate terms further to see that

$$\sum_{l=k}^{m}(\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})(|A_l|+|B_l|)\leq\sum_{l=1}^{m}(\frac{\delta\tau_l-1}{\tau_l+1})(|A_l|+|B_l|)+\sum_{l=1}^{k-1}(|A_l|+|B_l|)(\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1}).$$

As a consequence, one of the following must be true for the sum of these inequalities to be true:

$$\sum_{l=k}^{m}(\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})|B_l|\leq\sum_{l=1}^{m}(\frac{\delta\tau_l-1}{\tau_l+1})|A_l|+\sum_{l=1}^{k-1}|B_l|(\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1}) \tag{5.7}$$

$$\sum_{l=k}^{m}(\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})|A_l|\leq\sum_{l=1}^{m}(\frac{\delta\tau_l-1}{\tau_l+1})|B_l|+\sum_{l=1}^{k-1}|A_l|(\frac{\delta-\tau_{l+1}}{\tau_{l+1}+1}) \tag{5.8}$$

□

Therefore resolute single-winner voting rules which maintain Condition 5.4.1 can be created, and such a rule achieves optimal distortion between two candidates. We consider one such rule below, although many are possible.

**Weighted Majority Rule 4.**
*For all $l<k$, assign to all voters in $A_l$ and $B_l$ a weight of $\frac{(\delta+1)(\tau_l\tau_{l+1}-1)}{(\tau_l+1)(\tau_{l+1}+1)}$. For all $l\geq k$, assign voters in $A_l$ and $B_l$ a weight of $\left((\frac{\tau_{l+1}-\delta}{\tau_{l+1}-1})+(\frac{\delta\tau_l-1}{\tau_l+1})\right)$. Lastly, assign all voters in $C$ a weight of 0. Choose the candidate by a weighted majority vote.*

**Theorem 5.4.8.** *Weighted Majority Rule 4 satisfies Condition 5.4.1, and therefore achieves the optimal distortion for two candidates with preference strength information.*

*Proof.* Consider the two inequalities in Condition 5.4.1 which dictate whether it is permissible to choose $P$ or $Q$ respectively. We can take the difference RHS - LHS of each inequality, which must be non-negative for at least one of them, and choose the candidate corresponding to the inequality that yields a bigger difference. This is exactly our weighted majority rule. □

Weighted Majority Rule 4 is well-behaved because voters with weaker preferences are assigned smaller weights. Voters whose preferences are so weak that we cannot determine their preferred candidate must have a weight of 0 because it is unknown whom they support, and no voters have negative weight. However, voters with preference strength tending towards infinity cannot have infinitely large weights. Here, the weights of the voters whose decisiveness is higher than $\tau_m$ is $1 + \frac{\delta\tau_m - 1}{\tau_m + 1}$, which converges asymptotically to $\delta + 1$ as $\tau_m \to \infty$. However, many other rules with the same distortion are possible and it is an open question to determine which rules yield the best distortion for multiple candidates.

How much effort, time, and money, should someone charged with developing a voting protocol, or with choosing an alternative minimizing social cost, spend in order to understand the preference strengths of voters in more detail? With only ordinal preferences ($m = 1, \tau = 1$), the best distortion achievable is by simple majority vote, yielding a distortion of 3. However, if we are permitted any single threshold of our choice ($m = 1, 1 < \tau$), we can bring the distortion down significantly to 2. With any two thresholds of our choice ($m = 1, 1 \leq \tau_1 < \tau_2$), we can bring distortion down further to $5/3 \approx 1.67$, and as the number of thresholds permitted increases we see distortion converge to $\sqrt{2} \approx 1.4$. (See Figure 5.1.) This is because in the limit when we know the exact preference strengths of all voters, distortion can be bounded by $\sqrt{2}$, as we show in the next section. Thus, there is not much incentive to spend a huge amount of money to understand exact preference strengths, as one or two carefully chosen thresholds already provide very good distortion.

For the general case with arbitrary thresholds and no extra assumptions, we can demonstrate a bound of $\delta^2$ on the distortion for three or more candidates. This is obtained simply by forming a pairwise majority graph based on the above weighted majority rule, and then taking any alternative in the uncovered set of the resulting graph. It remains an open question whether there exist weighted majority rules that can improve the bound on distortion in the general case using this method, as we can when we have a single threshold and preferences, or preferences alone. More generally, it is unknown how to get a tight bound on the distortion with multiple candidates using any rule, even in the simpler case with only ordinal preferences [25].

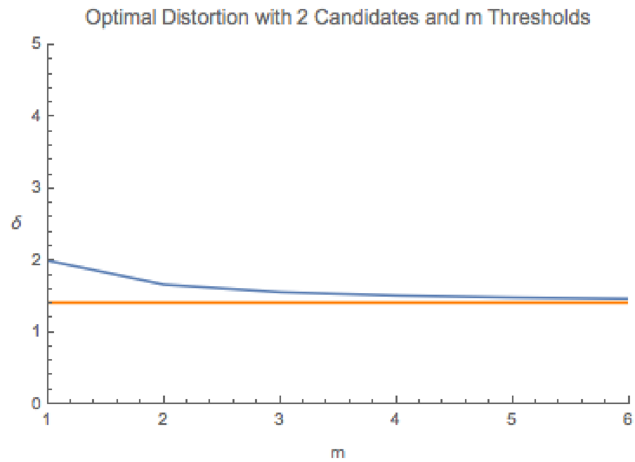


**Figure 5.2: Best achievable distortion for two candidates if allowed the best choice of $m$ thresholds. Converges to $\sqrt{2}$ with the number of thresholds.**

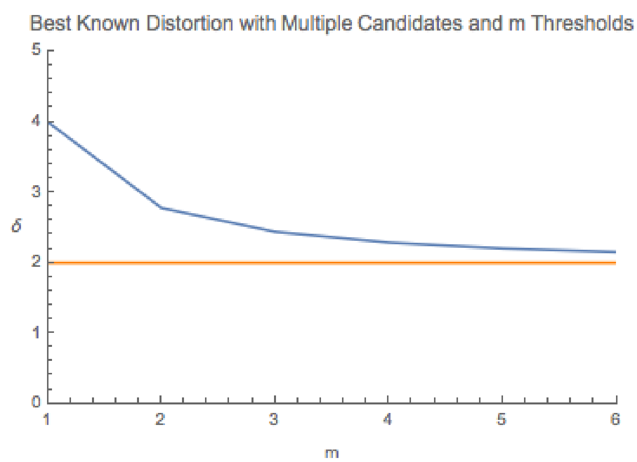


**Figure 5.3: Best known distortion for multiple candidates if allowed the best choice of $m$ thresholds. Converges to $2$ with the number of thresholds.**

### 5.4.1 Exact Preference Strengths of All Voters

In this section, for completeness of analysis, we consider the case when we know the exact preference strengths of all voters with respect to every pair of candidates. This corresponds to the limit settings in which we have an infinite number of thresholds that includes every number greater than 1. As we established previously, even with this knowledge it is not possible to form deterministic algorithms with distortion better than $\sqrt{2}$. Here we give a mechanism which obtains this bounds.

Suppose there are two candidates $P$ and $Q$, and we are given the preference strengths of every voter. Denote $A$ as the set of voters that prefer $P$ to $Q$, and $B$ as the set of voters that prefer $Q$ to $P$. The preference strength of any $i \in A$ is denoted as $\alpha_i$, and the preference strength of any $j \in B$ is denoted as $\beta_j$,

**Theorem 5.4.9.** *With 2 candidates $P$ and $Q$ in a metric, given the exact preference strength of every voter, if* $\sum_{i\in A} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} \geq \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\beta_j - \sqrt{2}}{\beta_j - 1} - \sum_{j\in B|\beta_j\leq\sqrt{2}} \frac{(\sqrt{2}-\beta_j)}{\beta_j+1}$*, then* $SC(P) \leq \sqrt{2}SC(Q)$.

*Proof.* $\forall i \in A$,

$$d(P,Q) \leq d(i,P) + d(i,Q) = \frac{\alpha_i + 1}{\alpha_i}(i,Q)$$

Bound the sum of $d(i,P)$ for all $i \in A$:

$$\begin{aligned}\sum_{i\in A} d(i,P) &= \sum_{i\in A} \frac{1}{\alpha_i} d(i,Q) \\ &= \sum_{i\in A} \frac{1}{\alpha_i} d(i,Q) + \sqrt{2}\sum_{i\in A} d(i,Q) - \sqrt{2}\sum_{i\in A} d(i,Q) \\ &= \sqrt{2}\sum_{i\in A} d(i,Q) - \sum_{i\in A}(\sqrt{2} - \frac{1}{\alpha_i})d(i,Q) \\ &\leq \sqrt{2}\sum_{i\in A} d(i,Q) - \sum_{i\in A}(\sqrt{2} - \frac{1}{\alpha_i})\frac{\alpha_i}{\alpha_i+1}d(P,Q) \\ &= \sqrt{2}\sum_{i\in A} d(i,Q) - \sum_{i\in A}\frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1}d(P,Q)\end{aligned}$$

We know that $\forall j \in B$ such that $\beta_j \leq \sqrt{2}$,

$$d(P,Q) \leq d(j,P) + d(j,Q) = (\beta_j + 1)(j,Q)$$

Bound the sum of $d(j,P)$ for all $j \in B$ that $\beta_j \leq \sqrt{2}$,

$$\begin{aligned}
\sum_{j\in B|\beta_j\le\sqrt{2}} d(j,P) &= \sum_{j\in B|\beta_j\le\sqrt{2}} \beta_j d(j,Q) \\
&= \sum_{j\in B|\beta_j\le\sqrt{2}} \beta_j d(j,Q) + \sqrt{2}\sum_{j\in B|\beta_j\le\sqrt{2}} d(j,Q) - \sqrt{2}\sum_{j\in B|\beta_j\le\sqrt{2}} d(j,Q) \\
&= \sqrt{2}\sum_{j\in B|\beta_j\le\sqrt{2}} d(j,Q) - \sum_{j\in B|\beta_j\le\sqrt{2}} (\sqrt{2}-\beta_j) d(j,Q) \\
&\le \sqrt{2}\sum_{j\in B|\beta_j\le\sqrt{2}} d(j,Q) - \sum_{j\in B|\beta_j\le\sqrt{2}} \frac{\sqrt{2}-\beta_j}{\beta_j+1} d(P,Q)
\end{aligned}$$

$\forall j \in B$ such that $\beta_j > \sqrt{2}$,

$$\begin{aligned}
d(j,P) &\le d(j,Q) + d(P,Q) \\
(1-\frac{1}{\beta_j})d(j,P) &\le d(P,Q) \\
d(j,P) &\le \frac{\beta_j}{\beta_j-1} d(P,Q)
\end{aligned}$$

We also know that $d(j,P) = \beta_j d(j,Q)$. Thus,

$$\begin{aligned}
d(j,P) &= \frac{\sqrt{2}}{\beta_j} d(j,P) + (1-\frac{\sqrt{2}}{\beta_j}) d(j,P) \\
&= \frac{\sqrt{2}}{\beta_j} \times \beta_j d(j,Q) + \frac{\beta_j-\sqrt{2}}{\beta_j} d(j,P) \\
&\le \sqrt{2} d(j,Q) + \frac{\beta_j-\sqrt{2}}{\beta_j} \times \frac{\beta_j}{\beta_j-1} d(P,Q) \\
&= \sqrt{2} d(j,Q) + \frac{\beta_j-\sqrt{2}}{\beta_j-1} d(P,Q)
\end{aligned}$$

Summing up for all $j \in B$ such that $\beta_j > \sqrt{2}$,

$$\sum_{j\in B|\beta_j>\sqrt{2}} d(j,P) \le \sqrt{2}\sum_{j\in B|\beta_j>\sqrt{2}} d(j,Q) + \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\beta_j-\sqrt{2}}{\beta_j-1} d(P,Q)$$

Putting everything together,

$$\sum_{i\in A} d(i,P) + \sum_{j\in B} d(j,P)$$
$$= \sum_{i\in A} \frac{1}{\alpha_i} d(i,Q) + \sum_{j\in B|\beta_j\le\sqrt{2}} d(j,P) + \sum_{j\in B|\beta_j>\sqrt{2}} d(j,P)$$
$$\le \sqrt{2} \sum_{i\in A\cup B} d(i,Q) + (-\sum_{i\in A} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} - \sum_{j\in B|\beta_j\le\sqrt{2}} \frac{\sqrt{2}-\beta_j}{\beta_j+1} + \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\beta_j-\sqrt{2}}{\beta_j-1}) d(P,Q)$$
$$\le \sqrt{2} \sum_{i\in A\cup B} d(i,Q)$$

□

**Weighted Majority Rule 5.** *Given the exact preference strength of every voter for two candidates, assign weight* $\frac{\sqrt{2}\alpha_i-1}{\alpha_i+1}$ *to each voter* $i \in A$ *such that* $\alpha_i > \sqrt{2}$*, and weight* $\alpha_i - 1$ *to each voter* $i \in A$ *such that* $\alpha_i \le \sqrt{2}$*. Assign weight* $\frac{\sqrt{2}\beta_j-1}{\beta_j+1}$ *to each voter* $j \in B$ *such that* $\beta_j > \sqrt{2}$ *and weight* $\beta_j - 1$ *to each voter* $j \in B$ *such that* $\beta_j \le \sqrt{2}$*.*

**Theorem 5.4.10.** *Using Weighted Majority Rule 5, the distortion is at most* $\sqrt{2}$ *for two candidates, and this is the best bound possible.*

*Proof.* Without loss of generality, suppose:

$$\sum_{i\in A|\alpha_i>\sqrt{2}} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i+1} + \sum_{i\in A|\alpha_i\le\sqrt{2}} (\alpha_i - 1) \ge \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\sqrt{2}\beta_j - 1}{\beta_j+1} + \sum_{j\in B|\beta_j\le\sqrt{2}} (\beta_j - 1)$$

And suppose we choose $P$ as the winner.

For $\alpha_i \le \sqrt{2}$, $(\alpha_i - 1) \le \frac{\sqrt{2}\alpha_i-1}{\alpha_i+1}$. By the condition above,

$$\sum_{i\in A|\alpha_i>\sqrt{2}} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} + \sum_{i\in A|\alpha_i\leq\sqrt{2}} (\alpha_i - 1) \geq \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\sqrt{2}\beta_j - 1}{\beta_j + 1} + \sum_{j\in B|\beta_j\leq\sqrt{2}} (\beta_j - 1)$$

$$\sum_{i\in A|\alpha_i>\sqrt{2}} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} + \sum_{i\in A|\alpha_i\leq\sqrt{2}} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} \geq \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\sqrt{2}\beta_j - 1}{\beta_j + 1} + \sum_{j\in B|\beta_j\leq\sqrt{2}} (\beta_j - 1)$$

$$\sum_{i\in A} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} \geq \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\beta_j - \sqrt{2}}{\beta_j - 1}$$

$$\sum_{i\in A} \frac{\sqrt{2}\alpha_i - 1}{\alpha_i + 1} \geq \sum_{j\in B|\beta_j>\sqrt{2}} \frac{\beta_j - \sqrt{2}}{\beta_j - 1} - \sum_{j\in B|\beta_j\leq\sqrt{2}} \frac{(\sqrt{2} - \beta_j)}{\beta_j + 1}$$

The second to last line follows because $\forall\beta_j \geq 1$, $\frac{\sqrt{2}\beta_j - 1}{\beta_j + 1} \geq \frac{\beta_j - \sqrt{2}}{\beta_j - 1}$. By Theorem 5.4.9, the distortion is at most $\sqrt{2}$.

Now we show the claim above that $\forall\beta_j \geq 1$, $\frac{\sqrt{2}\beta_j - 1}{\beta_j + 1} \geq \frac{\beta_j - \sqrt{2}}{\beta_j - 1}$ to finish the proof.

$$\begin{aligned}
(\beta_j - (\sqrt{2} + 1))^2 &\geq 0 \\
\beta_j^2 - 2(\sqrt{2} + 1)\beta_j + (\sqrt{2} + 1)^2 &\geq 0 \\
(\sqrt{2} - 1)\beta_j^2 - 2(\sqrt{2} + 1)(\sqrt{2} - 1)\beta_j + (\sqrt{2} + 1)^2(\sqrt{2} - 1) &\geq 0 \\
(\sqrt{2} - 1)\beta_j^2 - 2\beta_j + \sqrt{2} + 1 &\geq 0 \\
\sqrt{2}\beta_j^2 - \beta_j + 1 &\geq \beta_j^2 + \beta_j - \sqrt{2} \\
\sqrt{2}\beta_j^2 - \beta_j - \sqrt{2}\beta_j + 1 &\geq \beta_j^2 + \beta_j - \sqrt{2}\beta_j - \sqrt{2} \\
(\sqrt{2}\beta_j - 1)(\beta_j - 1) &\geq (\beta_j - \sqrt{2})(\beta_j + 1) \\
\frac{\sqrt{2}\beta_j - 1}{\beta_j + 1} &\geq \frac{\beta_j - \sqrt{2}}{\beta_j - 1}
\end{aligned}$$

□

**Corollary 5.4.10.1.** *Choosing a candidate from the uncovered set of a weighted majority graph obtained by using pairwise rule 5 results in distortion of at most 2 for any number of candidates.*

This corollary is simply because if pairwise distortion is at most $\delta$, then the distortion of the uncovered set is at most $\delta^2$. While for other special cases we have better bounds on the distortion with multiple candidates, for this case this general bound provides the best result.

## 5.5 Bad Examples and Lower Bounds

Note that our Algorithm 10 is only for social choice problems, and does not fit in the definition of our general facility assignment problems. This is because the median cost function, unlike sum and maximum, is not subadditive. In fact, while both min-cost and egalitarian matching problems have algorithms with small distortion in our setting, the same is not possible for forming a matching where the objective function is the cost of the *median* edge.

**Theorem 5.5.1.** *The worst-case distortion of the median-cost bipartite matching problem in a metric space (given both agent preference profiles and distances between facilities) is unbounded.*

*Proof.* Consider the following example: there are three agents $a$, $b$, $c$, and three facilities $X$, $Y$, $Z$. The preferences of the agents are: $a, b \in XYZ$, while $c \in ZXY$. The distances between facilities are: $\ell(X, Y) = 2$, $\ell(X, Z) = \ell(Y, Z) = 1000$. The distances between the agents and facilities are, of course, unknown. Consider the instance $d(c, Z) = \epsilon$, $d(a, X) = 2\epsilon$, and $d(b, X) = d(b, Y) = 1$. $\epsilon$ is a very small positive real number, and other distances not given obey triangle inequality. In this instance, the optimal solution is $x^* = \{(a, X), (b, Y), (c, Z)\}$, which gives a median value of $2\epsilon$. But because $a$ and $b$ have the same preference profile, the instance could also be $d(c, Z) = \epsilon$, $d(b, X) = 2\epsilon$, and $d(a, X) = d(a, Y) = 1$. If we still return the assignment $x^*$ for this instance, the median would be 1. The distortion is arbitrarily bad when $\epsilon$ approaches 0. □

The following Theorems show some of the lower bounds mentioned in Table 1.2.

**Theorem 5.5.2.** *The worst-case distortion for the facility location problem in a metric space (given only agents' preference profiles) is unbounded.*

*Proof.* Consider the following example: there are two agents 1, 2, and two facilities $X$, $Y$. Agent 1 prefers $X$ to $Y$, while agent 2 prefers $Y$ to $X$. The opening costs are: $c_f(X) = 1$, $c_f(Y) = 100$. We can choose to open one facility or both of them.

**Case 1.** Suppose we only open $X$. Consider the following distances between the agents and facilities: $d(1, X) = d(2, Y) = 1$, $d(1, Y) = d(2, X) = L$, for some very large $\ell$. If we only open $X$, then the total cost is $> L$. While the optimal solution is to open both $X$ and $Y$, which has a total cost of 103. The distortion is unbounded.

**Case 2.** Suppose we only open $Y$. Consider the same distances as in **Case 1**, then the total cost is also $\ell$. And the optimal solution still has a total cost of 103. The distortion is unbounded.

**Case 3.** Suppose we open both facilities. Consider the following distances between the agents and facilities: $d(1, X) = d(1, Y) = d(2, X) = d(2, Y) = \epsilon$, where $\epsilon$ is a very small positive real number. If we open both facilities, the total cost is $101 + 2\epsilon$. While the optimal solution is to only open $X$ , which has a total cost of $1 + 2\epsilon$. If we increase $c_f(Y)$, the approximation ratio is unbounded. □

**Theorem 5.5.3.** *The worst-case distortion for the k-median problem in a metric space (given only agents' preference profiles) is at least* $\Omega(n)$.

*Proof.* Consider the following example: There are three facilities $X$, $Y$, and $Z$. There are $q$ agents who prefer $X$ to $Y$ to $Z$, $q$ agents who prefer $Y$ to $X$ to $Z$, and 1 agent who prefers $Z$ to $X$ to $Y$. We denote these three sets of agents as $\mathcal{A}_X$, $\mathcal{A}_Y$ and $\mathcal{A}_Z$ separately. Suppose $k = 2$, then we have three choices of the winners:

**Case 1.** Choose $X, Y$ as the winners. Consider the following distances between agents and facilities: $d(X, Y) = 1$, $d(Y, Z) = d(X, Z) = L$ for some very large $\ell$. $\mathcal{A}_X$ is located at the same location as $X$, $\mathcal{A}_Y$ is at the same location as $Y$, and $\mathcal{A}_Z$ is at the same location as $Z$. The cost of choosing $X, Y$ as the winners is $\ell$ because we need to assign the agent in $\mathcal{A}_Z$ to $X$ or $Y$. While the optimal solution is to choose $Y, Z$ as the winners, and get a total cost of $q$. So the distortion in this case is unbounded.

**Case 2.** Choose $X, Z$ as the winners. Consider the following distances between agents and facilities: $d(X, Y) = d(Y, Z) = d(X, Z) = 1$, and $\mathcal{A}_X$ locate on top of $X$, $\mathcal{A}_Y$ locate on top of $Y$, and $\mathcal{A}_Z$ locate on top of $Z$. The cost of choosing $X, Z$ as the winner is $q$, while the optimal solution is to choose $X, Y$ as the winners, and get a total cost of 1. The distortion is $q$ in this case.

**Case 3.** Choose $Y, Z$ as the winners. Consider the same distances as in **Case 2**. If we choose $Y, Z$ as the winners, the total cost is still $q$, and the distortion of this case is also $q$.

The total number of agents is $n = 2q + 1$, so we can conclude that the distortions in all these three cases are at least $\Omega(n)$. $\square$

**Theorem 5.5.4.** *The worst-case distortion of the egalitarian bipartite matching problem in a metric space (given only agents' preference profiles) is at least* 2.

*Proof.* Consider the following example: there are two agents 1, 2, and two facilities $X$, $Y$. Both agents prefer $X$ to $Y$. W.L.O.G., assume we match agent 1 to $X$, and agent 2 to $Y$. Suppose the distances between agents and facilities are: $d(1, X) = d(1, Y) = 1$, $d(2, X) = \epsilon$, $d(2, Y) = 2$, where $\epsilon$ is a very small positive real number. The egalitarian cost of our matching is 2, while the optimal solution is to match agent 1 to $Y$, and agent 2 to $X$, which has a cost of 1. $\square$

## 5.6 Conclusion

As we have shown, even a tiny amount of preference strength information allows us to significantly improve the distortion of social choice mechanisms. We quantify tradeoffs between the amount of information known about preference strengths and the achievable distortion and provide advice about which type of information about preference strengths seems to be the most useful.

When voters provide a single bit of extra preference strength information beyond their ordinal preferences, the distortion drops from 3 down to 1.83 between two candidates and from 4.236 down to 3.35 for multiple candidates if we can choose our threshold. When the exact preference strengths of all voters are known the distortion falls precipitously down to $\sqrt{2}$ for two candidates and 2 for multiple candidates. In general, with only one or two chosen thresholds, one would not choose a threshold of $\tau_1 = 1$, since it conveys less information than a slightly larger threshold. Intuitively, having a small barrier to voting that requires some effort to overcome means that only the votes of those with some stake in the outcome are included, but setting such a barrier too high can mean that many people with some interest in the decision are excluded. If we have more thresholds at our disposal we can further minimize distortion, but there are diminishing returns to additional thresholds. Considering the large improvements to distortion given just a single extra threshold, further information may not be worth the effort to obtain.

# CHAPTER 6
# Future Directions

In this thesis, we study ordinal approximation of matching, social choice and other problems in a metric space. The distances between agents (or candidates/facilities) represents agents' costs or utilities. We consider different levels of ordinal information (one-sided, two-sided, total ordering) for the maximum weight bipartite matching problem, and various objectives (total, median, and egalitarian social cost) for the social choice and other problems in Chapter 4, when we know agents' ordinal preferences and candidates' exact locations. Many open questions remain in our settings. What if we could obtain some other information, but at further costs? What about randomized mechanisms for the model that we know the location of candidates, or what if the mechanisms must be truthful? And more generally, exactly what information is enough to guarantee mechanisms with small distortion? We discuss future direction for related ordinal approximation problems in a metric space in this chapter.

## 6.1 Bipartite Matching

1. **Upper bound of RSD on maximum weight bipartite matching.** From Chapter 3, we know that the current lower bound of RSD is 1.62, while the upper bound is 2.41. An immediate direction is to close the gap between the upper and lower bound. We have results showing that RSD has a lower upper bound in some special cases. For example, consider a bipartite graph that all the agents in $\mathcal{X}$ prefer $y_1$ to $y_2$, $\ldots$ to $y_N$. For any agent $x_i$ when $i \leq kN$, $k$ could be any value $\in [0,1]$, $\forall j \leq i,\ w(x_i, y_j) = 3$, and $\forall j > i,\ w(x_i, y_j) = 1$. All other edges' weight are 1. In this special case, we can show that the upper bound of RSD is 2.

   In order to get a better lower bound example, we generate random bipartite graphs that obey the metric space requirements, then estimate the approximation ratio of RSD by random sampling the order of agents in $\mathcal{X}$ for a large number of rounds. The highest approximation example we found has an estimated RSD approximation ratio of 2.03 by running the random sampling for 100000 rounds. Note that these are only estimations of RSD on random samples, but the results indicate that the upper and lower bound of the approximation ratio of RSD could be 2.

2. **Better algorithms for the two-sided and total ordering model on maximum weight bipartite matching.** Similar to the one-sided model, the uppper bound and lower bound given in Chapter 3 are not tight. One reason is that we only consider bipartite graphs that only have edges of weights 1 and 3 to construct the lower bound examples, to make sure the graphs are in a metric space. And we might need a different algorithm to achieve a better upper bound.

3. **Truthful algorithms for the two sided and total ordering models.** For the one sided model, we analyzed the approximation of RSD, which is truthful. However, the algorithms for the two sided and total ordering models are not truthful. Of course, we can also run RSD on them, but the bound might be loose. Anshelevich et al. [19] study truthful algorithms for non-bipartite maximum weight matching and other problems. Are there also better truthful algorithms for these two models?

4. **Augmentation for the maximum weight bipartite matching.** Instead of bipartite matching, what if we consider the facility assignment problem with augmentation, which means instead of matching to only 1 agent, each facility has an augmented capacity of $g$. If we compare the performance of SD on the augmented model with the optimal solution of the matching problem without augmentation, we can show that SD has a $\frac{g}{g+2}$-approximation to the optimal solution. If we compare SD on the augmented model with the optimal solution to the same model, we know SD has a 3-approximation to the optimal solution. Are there other mechanism that would give better approximation for this model?

5. **Minimum weight bipartite matching.** The best known algorithm for minimum weight bipartite matching is given by Caragiannis et al. [5]. They showed that approximation ratio of random serial dictatorship (RSD) is at most $n$. While the current known lower bound is 3, there is still a huge gap between the upper and lower bound.

   We considered some speical cases of this problem. It is easy to show that the distortion of SD is at most 3 when every agent in $\mathcal{X}$ has the same top choice agent in $\mathcal{Y}$. We also tried to propose a multiple step algorithm: first get the set of agents in $\mathcal{Y}$ that is the top choice of some agents in $\mathcal{X}$, match each one in this set of "top agents in $\mathcal{Y}$" with an arbitrary agent in $\mathcal{X}$ that prefer it the most. If there are unmatched agents, repeat this process with the set of "top agents" in the unmatched ones in $\mathcal{Y}$ until all agents are matched. For this multiple step greedy algorithm, we know that if the process ends within two rounds, the distortion is at most 3, and if ends within three rounds, the distortion is at most 5. And both bounds are tight for this algorithm.

Can we give a lower distortion in some other special cases, e.g. all agents are on a line, or in a tree structure?

6. **Augmentation for the minimum weight bipartite matching.** Caragiannis et al. [5] study the performance of SD and RSD for the minimum weight bipartite matching with augmentation. We also consider this problem in some special cases. Suppose the capacity of each agent in $\mathcal{Y}$ is $g$ instead of 1. We consider the special case that all the agents in $\mathcal{X}$ have the same preferences over the agents in $\mathcal{Y}$, and showed that SD on the augmented model has a approximation ratio of at most $(1 + \frac{2}{g^2-g+1})$ compared to the optimal solution of the original model.

Another clear research direction is to relax the assumption that we can only obtain ordinal information. What if we could also obtain some numerical information, but at further cost? What is the tradeoff between quality of solution formed and the amount of numerical information we obtain? What if we could ask the agents more complex questions than "Who is your favorite unmatched agent?", but were limited in the number of times we could ask such questions? For example, we can give prices for each agent, and ask "Given these prices, what is your favorite agent?", or "Tell me all agents that has value more than 100 for you.".

## 6.2 Social Choice

In Chapter 4, we study the setting with agents' ordinal preferences over all candidates and the exact locations of the candidates. Our deterministic algorithms give a tight distortion of 3 for both the minimum and the median social cost problems in this setting. In Chapter 5, we study social choice with agents' ordinal preferences and information of their preference strength on the candidates. There are also a lot of open questions in the pure ordinal model and other settings with different information.

1. **Randomized algorithms with ordinal preferences and the location of the candidates.** We give a deterministic algorithm that has a tight distortion of 3 in

this setting in Chapter 4. How about the randomized algorithm? The trivial lower bound example for the deterministic algorithm also gives a lower bound of 2 for the randomized algorithms. Anshelevich et al. [12] study the randomized algorithm in the pure ordinal setting, and show that random dictatorship gives a tight distortion of at most 3. Because the lower bound example of random dictatorship only has two candidates, so knowing the exact locations of the candidates would not help if we use "weighted" random dictatorship based on the distances among the candidates. But are there other mechanisms to use this extra information?

2. **Truthful algorithms with ordinal preferences and the location of the candidates.** Our deterministic algorithm is not truthful. Can we design truthful algorithms? Would the truthful constraint make the distortion worse, and by how much?

3. **Algorithms satisfy multiple objective at the same time.** The greedy algorithm for the median objective in Chapter 4 actually gives a distortion of at most 3 for both the median and total social cost objective. Can we design algorithms that works for more than one objectives at the same time on other problems/objectives?

4. **The best deterministic social choice algorithm in the pure ordinal setting.** The best known deterministic algorithm for social choice gives a distortion of at most 4.236, while the best known lower bound is 3 [11],[25].

5. **The best randomized social choice algorithm in the pure ordinal setting.** The best known deterministic algorithm for social choice is Randomized Dictatorship, which gives a distortion of at most 3, while the best known lower bound is 2 [12],[15].

6. **The best deterministic social choice algorithm with preference strength information.** In Chapter 5 we study social choice given some information about voters' preference strength (different number of thresholds). Our results give the best possible distortion when there are only two candidates. However, if there are more than two candidates, there are still gaps between our results and the lower bounds.

7. **Randomized social choice algorithms with preference strenghth information.** We study deterministic voting rules with information about preference strengths in In Chapter 5. How to use these extra information in randomized algorithms? Are

we also able to improve the distortion a lot using randomized algorithms with these information?

8. **Move agents or candidates to get a good (or the optimal) winner.** Consider a certain voting rule, for any instance, can we only move a few agents or candidate to make sure that the algorithm selects the optimal candidate, or a candidate with low distortion? The constraint could also be: the total distances of the movements are small. For example, suppose there are only two candidates, and we select the one with more voters prefer it to be the winner. The worst case distortion is 3 in this case. However, in the worst case example, we only need to move a very small portion of the voters to make the true optimal candidate the winner.

9. **Special cases of agents' ordinal preferences.** What if (almost) all the agents have the same preferences over the candidates? Or only a few candidates are some agents' top choice?

10. **Existence of the Nash Equilibria (NE) and PoA, PoS.** For the non-truthful voting rules and our algorithm with ordinal preferences and the location of the candidates, does NE always exist? If the answer is yes, what is the PoA and PoS? For most of the voting rules, it is trivial that NE always exist if we put all the voters on top of one candidate, and the PoA would be arbitrarily bad in that case. So it is reasonable to assume the voters are truth bias, which means the voters prefer to tell the truth if it does not affect the result. We can also assume that agents need to pay some money to lie. Unfortunately NE does not always exist in the truth bias setting, but how about approximate Nash Equilibria? In the setting that the voters are not truth-bias, strong equilibrium doesn't always exists, but there is always a 2-approximate strong equilibrium that has a cost no more than 4.236 times the optimal solution.

There are also many other interesting open questions about what information is enough to guarantee mechanisms with small distortion. Generally, agents give signals related to their distances to the candidates. Ordinal preferences is one type of such signals, for example: "These are all the candidates within a certain distance from me: ...". So far, we have been assuming agents' costs only depends on their distances to the candidates, but in real life, people's opinion are often greatly affected by their friends and family. What is a suitable model to represent this scenario?

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