2D LOG-ELASTOGRAPHIC METHODS
FOR TISSUE SHEAR STIFFNESS RECONSTRUCTION
USING A 2D PLANE STRAIN ELASTIC SYSTEM

By

Ning Zhang

A Thesis Submitted to the Graduate
Faculty of Rensselaer Polytechnic Institute
in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
Major Subject: Mathematics

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Rensselaer Polytechnic Institute
Troy, New York

August 2009
(For Graduation December 2009)
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ABSTRACT

As a promising medical imaging technique for disease diagnostics, elastography targets reconstructing and imaging elastic parameters in biological tissue, due to the fact that abnormal tissue and healthy tissue exhibit distinct elastic properties. In all the experiments developed so far, tissue is mechanically excited and the interior displacement of the propagating wave is measured using ultrasound or magnetic resonance imaging (MRI) in order to recover the elastic properties of the tissue.

This thesis concerns the reconstruction of shear stiffness biomechanical parameters. With given single frequency 2D elastographic displacement data, the mathematical model is a first order partial differential equation system derived from a 2D plane strain elastic system. A nonlinear 2D Log-Elastographic algorithm is developed to recover the shear modulus, together with the hydrostatic pressure, a term neglected in common practice when biomechanical parameters are imaged. In many previous works, the elastic system is reduced to either a Helmholtz equation or a single first order p.d.e., while in this work we utilize the 2D plane strain elastic system without neglecting the hydrostatic pressure term as the basis for our first order p.d.e. system.

The main advantage of the 2D Log-Elastographic method we develop here for solving the first order p.d.e. system is that it effectively controls possible exponential error growth without using a very fine discretization, a restriction needed by standard numerical methods. Stability and first order convergence are established.

Numerical examples with synthetic data show that the 2D Log-Elastographic algorithm improves the quality of the recovered images compared with the images obtained from the Direct Inversion method, the acoustic Log-Elastographic algorithm and the standard upwind algorithm. We also demonstrate that: (1) neglecting the hydrostatic pressure term can cause significant undershooting in the biomechanical image; and (2) we obtain agreement between the hydrostatic pressure obtained from this new algorithm and the hydrostatic pressure calculated with our forward algorithm.
Finally, images of stiffness variations in a diseased human liver are obtained by applying this 2D Log-Elastographic algorithm with *in-vivo* data from Richard Ehman’s laboratory at Mayo Clinic. In this case, an additional filtering step is added.
CHAPTER 1
Introduction

1.1 Problem Overview

Elastography is a newly developed non-invasive medical imaging technology which targets detection or classification of abnormal objects inside the human body, such as lesions, tumors and cancers, by their biomechanical properties. In Elastography, the shear properties of human tissue are chosen to image, as the expected contrast between normal tissue and abnormal tissue is high. This expected contrast drives the target which is high resolution images of tissue changes. The expectation is to identify and classify tissue abnormalities and to corroborate with ultrasound or MR images enhancing medical diagnosis by imaging the elastic properties of the tissue.

To obtain tissue displacement data to be used to image the shear properties, a broad range of experiments have been developed. These experiments are: (1) static experiment, e.g. see [50] and [59]; (2) dynamic sinusoidal excitation, e.g. see [8], [24], [28], [33], [34], [35], [40], [63], [66]; (3) acoustic radiation force impulse, e.g. see [15], [43], [44]; and (4) transient elastography and supersonic imaging, e.g. see [4], [5], [6], [7], [9], [22], [30], [57] and [58]. The common steps for all these procedures are as follows: first, apply an external or internal stimulus to excite the region of interest; then a movie of the interior displacement is created from successive ultrasound or MR images; the movie of the interior displacement is then the data used to create biomechanical property images; the displacement represented in these movies is on the order of tens of microns; with the measured displacement data, an inversion algorithm is developed to obtain tissue shear wave speed or shear modulus images. In this thesis, we will focus on the transient elastography and the dynamic sinusoidal excitation cases, where a rich set of time-dependent displacement data is provided. Here we assume that either a single frequency excitation is applied or single frequency content is extracted from the time dependent movie of displacement. Because the displacement deformations are small, the mathematical model we use
for describing these experiments is linear.

In the transient elastography experiments, tissue was initially at rest. Then a time-dependent pulse at the boundary or along a line inside the interior is initiated, creating two different types of waves that propagate into the interior region. The typical central frequency of the pulse is 50-200 Hz. The two types of propagating waves are compression waves and shear waves. The experiment is designed so that the compression wave has low amplitude. The displacement, which is then primarily a shear wave displacement, is then measured using ultrasound or MRI. These measured displacement data sets are used to find the wave speed of a propagating front, to find phase wave speed, or to use single frequency content for a shear modulus recovery.

In the dynamic sinusoidal excitation experiments, a time-harmonic excitation is made on the boundary, creating a time-harmonic wave. The data are recorded after the tissue has reached steady state. There are generally two measurement techniques for this problem. The first one is sonocelastography where real-time Doppler techniques are used to image the maximal amplitude of the harmonic displacement, see [22], [63] and [66]. The second one is magnetic resonance elastography (MRE). In MRE, the displacement of the propagating wave is measured as a function of space and time using MRI. Single frequency displacement content is derived from the data sets to recover the shear modulus, see [8], [24], [28], [33], [34] and [35].

A related experiment is the crawling wave experiment where two time-harmonic excitations at slightly different frequencies are created on opposite sides of the region of interest. These vibration sources produce a crawling wave, which moves very slowly and can be sampled by a Doppler ultrasound scanner at a frame rate similar to the ultrafast imaging system frame rate for a wave moving at the shear wave speed, see [67] and [68]. To date the data is spectral variance data. Although this experiment is related to the single frequency excitation experiment the spectral variance data is sufficiently different so that the results in this thesis will not be applied to this particular case.

However, the algorithm presented in this thesis is applied to MRE data from a single frequency excitation of an in-vivo human liver. Liver disease is on the increase,
affecting one out of every 10 Americans. Recently single and multifrequency excitation for MR Elastography and transient elastography have been applied to the study of liver fibrosis and cirrhosis, see [2], [10], [11], [18], [19], [26], [54], [55], [69], [70] and [71]. The MRE elastic displacement data enables a quantitative method to assess the mechanical properties of the liver tissues. Shear stiffness images obtained using this data at a single frequency have shown that a shear stiffness imaging functional in liver tissue increases systematically with onset of hepatic fibrosis and cirrhosis, [69] and [70]. In [2] and [26], an experiment producing multifrequency shear waves is performed and five standard rheological models (Maxwell, Voigt, Zener, Jeffreys and fractional Zener model) are assessed for their ability to reproduce the observed dispersion curves. Furthermore, shear modulus, shear wave speed and viscosity parameter are imaged to show that these quantities in fibrotic liver are much higher than in normal liver. We will compare our biomechanical property images produced with the \textit{in-vivo} human liver MRE data at a single frequency with both of their results in Chapter 6.

To reconstruct and image the shear wave speed or the shear modulus, different inversion methods have been developed. An often used method is the so-called Direct Inversion method, where a locally constant assumption for the shear modulus is made and the hydrostatic pressure is neglected. The elastic equation system is hence decoupled and reduced to a Helmholtz equation for individual displacement components. The shear modulus can then be recovered through a simple algebraic inversion, see [4], [5], [6], [33], [34], [35], [36], [39], [48], [49] and [58]. This approach is very straightforward to implement. Some good reconstruction results have been obtained, see [39]. In [31], the authors establish a theoretical bound on the relative difference between the true value of the shear modulus and the approximated value reconstructed from the Direct Inversion where the underlying model is the acoustic equation. This bound shows that the locally constant assumption on the shear modulus may introduce a certain level of inaccuracy; especially for low frequencies and in regions where the gradient of the shear modulus is large, e.g., near the boundary of stiff inclusions. We will make numerical comparisons of reconstructions from this Direct Inversion method with reconstructions from our first order p.d.e.
system in a later chapter of this thesis. Another kind of Direct Inversion method, see [61] and [62], is to take curl of the elastic equation system first, and then neglect all the derivatives of the shear modulus. In 2D the reconstruction is then based on a single equation. In 3D three distinct algebraic equations arise from this procedure. The main disadvantages of this method are that: (1) third-order spatial derivatives of the measured displacement data must be calculated; and (2) both first and second order derivatives of the shear modulus are neglected.

In [37] and [38], the authors developed the Arrival Time algorithm for the transient elastography experiment and supersonic imaging to recover the shear wave speed using the front of the propagating shear wave. There, they first use a cross-correlation technique to find the arrival time surface from the displacement data. The arrival time surface satisfies the Eikonal equation. They then apply either the distance method or the level set method to solve the inverse Eikonal equation to find the wave speed. This new technique has also been successfully used in the crawling wave experiment, see [40]. Note also that adaptations of this idea are used in the time of flight algorithms employed in [9], [43], [44], [57] and [58].

Besides the Direct Inversion method and the Arrival Time method, other algorithms have also been developed. Some are based on finite element methods including iterative methods, [14], [45], [46], [47], [64] and [65], and non-iterative methods [51]. The finite element based iterative methods use the full elastic equation system as the basis for their algorithm, and the goal is to find the elastic modulus that minimizes the difference between the computed and measured displacements in a least squares sense. Because they require the full forward calculations and contain iterations, all the iterative methods are time-consuming. Furthermore they often require knowledge of boundary conditions for the displacement solution of the equations of elasticity. However, a gradient-based (quasi-Newton) optimization approach is used in [45], [46] and [47] to do the minimization where they calculate the gradient using the solution of the adjoint elasticity equations. This approach saves computational time significantly. But they also need boundary conditions. In [51], the authors developed a non-iterative finite element based method. Their method is also based on the full elastic equation system. There they use the weak form of the elastic
system, and then only the first order derivatives of the measured displacement data are calculated. They then solve the remaining system for the shear modulus and pressure with the finite element method.

Another non-iterative variational method, that has similarities with the Direct Inversion method in that it neglects derivatives of the shear modulus, is developed in [52] and [53]. This non-iterative variational method is obtained by integrating the weak form of the elastic system by parts twice, and hence avoids taking derivatives of the measured displacements; however all first derivatives of the shear modulus are neglected.

In [32], the authors assume that a single component of the measured displacement satisfies an acoustic wave equation and develop an acoustic Log-Elastographic nonlinear marching scheme and a linear finite difference based elliptic scheme to reconstruct a real shear modulus by solving a single first order p.d.e.. They show that both methods are stable and convergent at first order and do not require a very fine discretization to succeed. Furthermore, the authors use the Direct Inversion result for the needed boundary data and establish an error bound of the results from this choice.

In this thesis, we focus on using the full plane strain elastic equation system in $2D$ as the basis for our inversion algorithm. The data is two components of displacement. One of the difficulties with using this equation system is that it contains the hydrostatic pressure which cannot be determined directly from measurements. And so, in our algorithm, we develop a finite difference based inversion method, the $2D$ Log-Elastographic method, in which we reconstruct both the unknown shear modulus and the unknown hydrostatic pressure with the measured displacement data. Since we solve a first order p.d.e. system, we must select good approximate values for the boundary conditions for the shear modulus. We make our choice from one of the available direct inversion results. This choice is described in more detail in the thesis.

In the rest of this section, we give the mathematical model for the forward and inverse problem driving the results in this thesis. We also briefly describe the Direct Inversion methods, the acoustic Log-Elastographic method and the motivation for
the 2D Log-Elastographic algorithm.

1.2 Mathematical Model

Our target here is an appropriate 2D approximate model for the MRE experiment. The reason for using a 2D model is that we target experiments that produce two or three displacement components in a single image plane. The restriction to single image plane data can occur when it is difficult to remove motion artifacts, e.g. breathing motion artifacts, to obtain multiplane data. The medium is inherently 3D and is assumed to be heterogeneous, isotropic and nearly incompressible. The 3D mathematical model for these kinds of problems in cases where the tissue is inherently isotropic, as in liver, is based on the following linear isotropic hyperbolic elastic system:

\[
\rho u_{tt} = \nabla (\lambda \nabla \cdot u) + \nabla \cdot (\mu (\nabla u + \nabla u^T)) + f \tag{1.1}
\]

where \( u \) is the 3D displacement vector, \( \mu \) is the shear modulus, or second Lamé parameter, \( \rho \) is the density, which in tissue is assumed to be constant, \( \lambda \) is the first Lamé parameter, and \( f \) is a source interior to the material. We assume that either there is an impulsive force, with a central frequency, \( \omega_c \), or there is a single frequency excitation.

If we let \( p = \lambda \nabla \cdot u \), we obtain the equation system

\[
\rho u_{tt} = \nabla p + \nabla \cdot (\mu (\nabla u + \nabla u^T)) + f. \tag{1.2}
\]

Note that \( p \) is essentially the hydrostatic stress, or the hydrostatic pressure. To simplify our notation, we divide both sides of equation (1.2) by \( \rho \), which is assumed to be constant in tissue, and make the following notation: \( \tilde{\mu} = \mu / \rho \), \( \tilde{p} = p / \rho \), and \( \tilde{f} = f / \rho \), so that our model for the forward problem is: given \( \tilde{\mu} \) and \( \tilde{f} \), find \( u \) and \( \tilde{p} \) satisfying

\[
\begin{align*}
    u_{tt} &= \nabla \tilde{p} + \nabla \cdot (\tilde{\mu} (\nabla u + \nabla u^T)) + \tilde{f} \\
    \tilde{p} &= (\lambda / \rho) \nabla \cdot u. \tag{1.3}
\end{align*}
\]
To obtain the 2D model, we recall that from Hooke’s law, \( \sigma = 2\mu \epsilon + \lambda \text{tr}(\epsilon)I \), where \( \sigma \) is the stress, \( \epsilon \) is the strain tensor, defined by the formula \( \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \), \( I \) is the identity matrix and \( \text{tr}(\cdot) \) is the trace of the matrix. Here, to reduce our equation system to a system of two equations in two elastic components, \( u_1, u_2 \), we assume that \( \epsilon_{i,3} = 0 \), for \( i = 1, 2, 3 \), i.e.,

\[
u \epsilon_{1,3} + u_{3,1} = 0, \quad u_{2,3} + u_{3,2} = 0, \quad \text{and} \quad u_{3,3} = 0.
\]

This is the plane strain assumption. Furthermore, we ignore the third equation in (1.2) and obtain a 2D plane strain elastic system, which has the same format as (1.3) if we use the dual notation that \( u \) is the 2D displacement and \( \tilde{p} \) is the scaled pressure.

In the forward problem, the scaled shear modulus \( \tilde{\mu} \), the density \( \rho \), the first Lamé parameter \( \lambda \) and the source function \( f \) are all given. Then the above 2D elastic system (1.3) is solved for the displacement vector \( u \) and the pressure \( p \). While in the inverse problem, the Fourier transformed displacement vector \( \hat{u} \) is given, the density \( \rho \) is given as \( 10^3 \text{ kg/m}^3 \), and the goal is to image \( \tilde{\mu} \).

To achieve the goal of imaging \( \tilde{\mu} \) from \( \hat{u} \), we need an equation system satisfied by \( \tilde{\mu} \) and \( \hat{u} \). To obtain this equation system, we take the Fourier transform of the system (1.3) and arrive at the following Fourier transformed system at a single frequency, where \( \hat{u}, \hat{\tilde{p}}, \) and \( \tilde{f} \) are the transformed values of \( u, \tilde{p}, \) and \( f \),

\[
\nabla \cdot (\tilde{\mu} (\nabla \hat{u} + \nabla \hat{u}^T)) + \omega_c^2 \hat{u} + \nabla \hat{\tilde{p}} = 0
\]

\[
\hat{\tilde{p}} = (\lambda/\rho) \nabla \cdot \hat{u}
\]

(1.4)

where we have chosen the central frequency \( \omega_c \). In the following chapters, to solve the inverse problem we will not use the equation \( \hat{\tilde{p}} = (\lambda/\rho) \nabla \cdot \hat{u} \). The reason for this is that tissue is mostly fluid and is nearly incompressible so the volumetric change \( \nabla \cdot \hat{u} \ll 1 \). Furthermore, \( \lambda/\rho \approx (1500)^2 \text{ m}^2/\text{s}^2 \). The product \( (\lambda/\rho) \nabla \cdot \hat{u} \) is unreliable as computed from measured data. Hence when we develop our 2D Log-Elastographic marching algorithm to solve the inverse problem we use the remaining system in (1.4) to solve for both the scaled shear modulus, \( \tilde{\mu} \), and the scaled hydro-
static pressure $\hat{p}$ simultaneously with the displacement vector $\hat{u}$ given.

All of our synthetic examples are for real $\tilde{\mu}$. In this case equation (1.3) is valid. For our in-vivo data, a more accurate model includes a viscoelastic term. Here, for the viscoelastic model we could use the linear solid model which is

$$\rho u_{tt} = \nabla \cdot \sigma' + \nabla p + f$$

$$p = \lambda \nabla \cdot u$$

where

$$\sigma' = 2\mu_\infty \epsilon_{ij} + 2 \int_{t_0}^t \sum_{\alpha=1}^N \int_{t'} e^{-(t-t')/\tau_\alpha} \frac{d}{dt'} (\mu_\alpha \epsilon_{ij}) dt'$$

where $\mu_\infty$, $\mu_\alpha$ are shear moduli and $\tau_\alpha = \eta_\alpha/\mu_\alpha$ are the relaxation times, $\alpha = 1, ..., n$. When we take the Fourier transform for this model we obtain equation (1.4) again with the property that here, for this model,

$$\tilde{\mu} = \text{Re} \tilde{\mu} + i \text{Im} \tilde{\mu} = \frac{\mu_\infty}{\rho} + \sum_{\alpha=1}^n \frac{\omega^2 \mu_\alpha \tau_\alpha^2}{\rho(1 + \omega^2 \tau_\alpha^2)} + i \sum_{\alpha=1}^n \frac{\mu_\alpha \omega \tau_\alpha}{\rho(1 + \omega^2 \tau_\alpha^2)}$$

which is a complex quantity. The coefficients $\eta_\alpha$, $\mu_\alpha$ are the viscosity and elasticity, respectively, of a Maxwell body, i.e., the coefficients corresponding to an element that contains both a dashpot and a spring. The coefficient $\mu_0$ is the coefficient for the spring element; the spring element and all the individual Maxwell elements are in parallel ([16]). In the linear solid model, local wave speed squared becomes, see [17], $(\text{Re} \sqrt{1/\tilde{\mu}})^{-2}$.

### 1.3 Direct Inversion Methods for the Elastic System and Acoustic Log-Elastographic Method for the Acoustic Equation

Before introducing our 2D Log-Elastographic algorithm, we will give a brief description of three other inversion methods. We will compare the images obtained with these methods to the images obtained with the 2D Log-Elastographic method.
in the numerical experiments section.

1.3.1 Direct Inversion Method (I) Using the Locally Constant Assumption

The Direct Inversion method is a widely used inversion method because it is essentially an algebraic formula for the shear modulus and is straightforward to implement. For Method (I), the shear modulus is assumed to be locally constant, and hence all the terms that contain the derivatives of the shear modulus are neglected. Furthermore, the hydrostatic pressure is assumed to be constant so that $\nabla p$ is set equal to zero. With these assumptions, the 2D elastic system can be decoupled. After taking the Fourier transform in time, a single Helmholtz equation is obtained for each elastic component and at each frequency as follows:

$$\hat{\mu} \Delta \hat{u} + \omega^2 \hat{u} = 0 \quad (1.5)$$

where $\hat{u}$ is the Fourier transform of any single component of the previous displacement vector $u$, and we are denoting the direct inversion approximation of $\tilde{\mu}$ by $\hat{\mu}$. The method then is to solve this algebraic equation for the approximate scaled shear modulus $\hat{\mu}$. The frequency, $\omega$, is usually chosen to be the central frequency, $\omega_c$.

1.3.2 Direct Inversion Method (II) by Taking Curl

Another often used direct inversion method is to first take curl of the elastic system. This has the benefit that $\text{curl}(\nabla p) = 0$. So that the term containing the pressure is eliminated. The resultant formula for $\tilde{\mu}$ is further simplified by neglecting all the terms containing derivatives of $\tilde{\mu}$, see [61] and [62]. In three dimensions, this procedure yields three separate equations of the form (1.5) where $\hat{u}$ is replaced by one of the three components of $\nabla \times u$. In two dimensions, when we take curl of the 2D elastic system, the following equation is obtained,

$$\tilde{\mu} \Delta (\nabla \times \hat{u}) + \tilde{\mu}_x((\nabla \times \hat{u})_x - \Delta \hat{u}_2) + \tilde{\mu}_y((\nabla \times \hat{u})_y + \Delta \hat{u}_1)$$
$$+ (\tilde{\mu}_{yy} - \tilde{\mu}_{xx})(\hat{u}_{1,y} + \hat{u}_{2,x}) + 2\tilde{\mu}_{xy}(\hat{u}_{1,x} - \hat{u}_{2,y}) + \omega_c^2 \nabla \times \hat{u} = 0$$
where $\nabla \times \hat{u}$ is a single function. Neglecting all the derivative terms of $\tilde{\mu}$ we obtain the single algebraic formula for

$$\hat{\mu}_c = -\omega^2_c (\nabla \times \hat{u}) / \Delta (\nabla \times \hat{u})$$

This is the 2D Direct Inversion approximation formula for the complex shear modulus obtained by taking curl. We denote this approximation by $\hat{\mu}_c$.

### 1.3.3 Acoustic Log-Elastographic Algorithm

If we don’t make the above locally constant assumption, and only neglect the pressure and $\nabla \cdot (\tilde{\mu} \nabla u^T)$ terms while keeping all the remaining $\nabla \tilde{\mu}$ terms, the 2D elastic system can also be decoupled. After taking the Fourier transform and keeping the central frequency content, an acoustic equation is obtained at single frequency as follows:

$$\tilde{\mu} \Delta \hat{u} + \nabla \tilde{\mu} \cdot \nabla \hat{u} + \omega^2_c \hat{u} = 0. \quad (1.6)$$

Here $\hat{u}$ represents any one of the displacement components. This is a single first order p.d.e.. The Acoustic Log-Elastographic algorithm is developed in [32] to find $\tilde{\mu}$ using (1.6). To describe the Acoustic Log-Elastographic algorithm, first we assume that $\hat{u}_x$ is nonzero. Then we can divide both sides of the above equation (1.6) by $\hat{u}_x$ and arrive at the equivalent first order p.d.e.

$$\tilde{\mu}_x + a \tilde{\mu}_y + b \tilde{\mu} + c = 0 \quad (1.7)$$

where $a = \hat{u}_y/\hat{u}_x$, $b = \Delta \hat{u}/\hat{u}_x$ and $c = \omega^2_c \hat{u}/\hat{u}_x$. The goal is to develop a new, but equivalent equation where possible exponential error can be controlled without using a very fine discretization. To achieve this, divide the equation (1.7) by $\tilde{\mu}$ to obtain

$$\tilde{\nu}_x + a \tilde{\nu}_y + b + ce^{-\bar{v}} = 0. \quad (1.8)$$
where $\tilde{\nu} = \log(\tilde{\mu})$. Assuming $\tilde{\nu}$ is real, the real part of equation (1.8) is discretized with a standard upwind scheme. Then the exponential of the discretized equation is taken to obtain another nonlinear discretized equation for $\tilde{\mu}$. The result is

$$
\tilde{\mu}_{i+1,j} = \tilde{\mu}_{i,j} e^{(1 - \text{sgn}(a_{i,j})a_{i,j} \Delta x) \frac{\Delta x}{\Delta y} - \text{sgn}(a_{i,j})a_{i,j} \Delta y \Delta x} e^{-(b_{i,j} + \frac{c_{i,j}}{\tilde{\mu}_{i,j}}) \Delta x}
$$

where $\tilde{\mu}_{i,j}, a_{i,j}, b_{i,j}$ and $c_{i,j}$ are the discretized values of $\tilde{\mu}, a, b$ and $c$ at the grid point $(x_i, y_j)$. This is the Acoustic Log-Elastographic algorithm for solving the acoustic equation (1.6) for $\tilde{\mu}$. The purpose of this algorithm is to control potential exponential growth in the numerical error that occur when the non-zero coefficient, Re$(b)$, of $\tilde{\mu}$ in (1.7) has the wrong sign. The term in the exponent is the direct inversion formula. The direct inversion formula is nearly zero when $\tilde{\mu}$ is slowly varying and this plays an important part in controlling possible exponential numerical error. The boundary conditions are chosen to be the results obtained from the direct inversion method and an error bound for the constructed $\tilde{\mu}$ due to this choice of boundary condition is established.

1.4 Motivation for and Difficulties in Developing the 2D Log-Elastographic Algorithm

First, numerical results show that the term $(\lambda/\rho) \nabla \cdot \mathbf{u}$, i.e., the scaled hydrostatic pressure, $\tilde{p}$, and its transform, $(\lambda/\rho) \nabla \cdot \hat{\mathbf{u}}$, are important inputs for reconstructing the shear modulus. In a nearly incompressible material, $\nabla \cdot \mathbf{u}$ is very small, but not equal to zero, and $\lambda/\rho$ is large, usually on the order of $10^6 m^2/s^2$. The product of those two values, i.e. the scaled pressure, can be computed to show that it is approximately first order and should not be ignored. Furthermore, up to now it has not been possible to measure hydrostatic pressure directly. Second, the bound obtained in [31] shows that at low frequencies and in regions where large gradients of $\tilde{\mu}$ occur, the derivative of $\tilde{\mu}$ terms should not be neglected. Third, it may not be possible to model the experiment so that the p.d.e. for one single component essentially decouples from the remaining system. On the positive side, in MRE it is often possible to measure two or all three components of the displacement data in
a single plane, and in this case the 2D plane strain elastic system is a more suitable mathematical model to solve in 2D. Therefore, in this thesis, we will focus on the 2D plane strain elastic system without neglecting any terms. Our main object is to solve this equation system simultaneously for the scaled shear modulus $\tilde{\mu}$ and the pressure $\hat{p}$ given the transformed displacements $\hat{u}$ throughout the plane. A finite difference based method, the 2D Log-Elastographic algorithm, will be presented to solve this equation system for $\tilde{\mu}$ and $\hat{p}$.

In developing the 2D Log-Elastographic algorithm we will overcome some difficulties. The main difficulties will be: (1) when coefficient matrices are real and have real eigenvalues, the solution to this equation system may have exponential growth if the eigenvalues of the coefficient matrix in front of the zeroth order derivative term of the shear modulus have wrong signs; in order to control this exponential growth, if no new steps are added, a very fine discretization needs to be applied; this will require additional memory and computational time; to overcome this problem we develop a nonlinear algorithm that can control this resultant possible exponential numerical error; and (2) when coefficient matrices are complex and have complex eigenvalues, the solution will have exponential growth of arbitrarily high order. However, in this case we demonstrate with in-vivo data that our 2D Log Elastographic algorithm, together with an additional filtering step, effectively controls this exponential growth without a fine grid requirement.

1.5 Remaining Thesis Structure

The remainder of this thesis is composed as follows. In Chapter 2 we establish the Reduced 2D Log-Elastographic method which is based on the reduced elastic equation system where the strain $\epsilon$ is replaced by $\nabla u/2$. Then, the 2D Log-Elastographic algorithm which is based on the elastic equation system is presented in Chapter 3. Next in Chapter 4 we establish the stability and accuracy results for the 2D Log-Elastographic algorithm under the assumption that eigenvalues of coefficient matrix are real. Then in Chapter 5 and Chapter 6, this new algorithm is tested on both synthetic data and experimental data. Finally, we give some conclusions and discussions in Chapter 7.
CHAPTER 2
Inversion Algorithm Using the Reduced System

As mentioned in Chapter 1, our long range goal is to use equation (1.3), but as a first step, we consider a reduced equation. In this step, we show the underlying ideas of our algorithm. In Chapter 3, we consider the full 2D plane strain elastic system. So for this step we neglect the term \( \nabla \cdot (\tilde{\mu} \nabla u^T) \) in (1.3) and arrive at the reduced equation

\[
u_{tt} = \nabla \cdot (\tilde{\mu} \nabla u) + \nabla \tilde{p} + \tilde{f}.
\tag{2.1}
\]

Our algorithm will be focused on the frequency domain case. Taking the Fourier transform of the above reduced equation, we start with the following equation at the main or central frequency \( \omega_c \):

\[
\nabla \cdot (\tilde{\mu} \nabla \hat{u}) + \nabla \hat{p} + \omega_c^2 \hat{u} = 0
\tag{2.2}
\]

where \( \hat{u}, \hat{p} \) are the Fourier transforms of \( u \) and \( \tilde{p} \). We also now set the forcing term \( \tilde{f} \) equal to zero since in our application \( \tilde{f} \) is nonzero only very close to the boundary. Our recoveries will not include the force location.

To see the structure of this first order system of equations, we rewrite (2.2) in matrix form:

\[
\begin{pmatrix}
\hat{u}_{1,x} & 1 \\
\hat{u}_{2,x} & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mu} \\
\hat{p}
\end{pmatrix}_x +
\begin{pmatrix}
\hat{u}_{1,y} & 0 \\
\hat{u}_{2,y} & 1
\end{pmatrix}
\begin{pmatrix}
\hat{\mu} \\
\hat{p}
\end{pmatrix}_y
+
\begin{pmatrix}
\Delta \hat{u}_1 & 0 \\
\Delta \hat{u}_2 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mu} \\
\hat{p}
\end{pmatrix} + \omega_c^2 \hat{u} = 0.
\tag{2.3}
\]

Denoting

\[
v := \begin{pmatrix} \hat{\mu} \\ \hat{p} \end{pmatrix}, \quad A := \begin{pmatrix} \hat{u}_{1,x} & 1 \\ \hat{u}_{2,x} & 0 \end{pmatrix},
\]

\]

13
$B := \begin{pmatrix} \hat{u}_{1,y} & 0 \\ \hat{u}_{2,y} & 1 \end{pmatrix}$, and $C := \begin{pmatrix} \Delta \hat{u}_1 & 0 \\ \Delta \hat{u}_2 & 0 \end{pmatrix}$,

then $A^{-1} = \begin{pmatrix} 0 & \frac{1}{\hat{u}_{2,x}} \\ 1 & -\frac{\hat{u}_{1,x}}{\hat{u}_{2,x}} \end{pmatrix}$. Multiplying $A^{-1}$ on both sides we obtain another system of first order equations of the form

$$v_x + E^1 v_y + E^2 v + E^3 = 0 \quad (2.4)$$

where

$$E^1 = \begin{pmatrix} \hat{u}_{2,y} & \frac{1}{\hat{u}_{2,x}} \\ \hat{u}_{1,y} - \frac{\hat{u}_{1,x}}{\hat{u}_{2,x}} \hat{u}_{2,y} & -\frac{\hat{u}_{1,x}}{\hat{u}_{2,x}} \end{pmatrix}, \quad E^2 = \begin{pmatrix} \Delta \hat{u}_2 & 0 \\ \Delta \hat{u}_1 - \frac{\hat{u}_{1,x}}{\hat{u}_{2,x}} \Delta \hat{u}_2 & 0 \end{pmatrix},$$

and $E^3 = \begin{pmatrix} \omega_c^2 \hat{u}_2 \\ \omega_c^2 (\hat{u}_1 - \frac{\hat{u}_{1,x}}{\hat{u}_{2,x}} \hat{u}_2) \end{pmatrix}$.

Here, to obtain the linear system (2.4) for the reduced equation, we selected one of the components $\hat{u}_1$ or $\hat{u}_2$ of $\hat{u}$, which we call the leading component. Usually it is the component that is larger, e.g. the component orthogonal to the direction of propagation. We also check the zero set of the derivative of that component in the direction of propagation as our algorithm requires that this derivative is nonzero. In the equation (2.4), $\hat{u}_2$ is the leading component and we have assumed that $\hat{u}_{2,x} \neq 0$. With this assumption, we can interpret, for the purpose of solving the first order p.d.e. system (2.4) with a marching method, that the $x$ direction is a time-like direction. In keeping with this interpretation we assume that $v(0, y)$ is known. Later in this document we will address how to get good approximations for $v(0, y)$.

### 2.1 Standard Partially Implicit Upwind and Stable Central Difference Schemes

First we apply the standard upwind scheme [12] to solve the above first order p.d.e. system (2.4) for $v$. In this case we need to decide the finite difference approximation upwind direction according to the sign of the eigenvalues of the coefficient
matrix $E^1$. If $E^1 = S \Lambda S^{-1}$ is the eigenvalue decomposition of $E^1$, then we write $E^1$ as $E^1 = E^{1,+} + E^{1,-}$, where $E^{1,+} = S \Lambda^+ S^{-1}$ and $E^{1,-} = S \Lambda^- S^{-1}$. The diagonal matrix $\Lambda^+$ (respectively $\Lambda^-$) has all those eigenvalues, with nonnegative (respectively negative) real part, of $E^1$ on its diagonal, and $S$ is the eigenvector matrix. Then the partially implicit upwind scheme for our system (2.4) is as follows:

$$V_{i,j}^{t+1} = V_{i,j}^t - \Delta x \left( \frac{E_{i,j}^{1,+} (V_{i,j}^t - V_{i,j-1}^t) + E_{i,j}^{1,-} (V_{i,j+1}^t - V_{i,j}^t)}{\Delta y} \right) - \Delta x E_{i,j}^2 V_{i+1,j}^t - \Delta x E_{i,j}^3$$

(2.5)

where $V_{i,j}$, $E_{i,j}^{1,+}$, $E_{i,j}^2$, and $E_{i,j}^3$ are the approximated values of $v$, $E^1$, $E^2$, and $E^3$, respectively, at the point $(i \Delta x, j \Delta y)$, with $V_{0,j}$ the known initial conditions.

We can also write a partially implicit scheme based on central difference approximations. Notice that what is important here is the representation for the $v_x$ derivative. Then the following central difference scheme for the system (2.4) results in a stable method, see [29]:

$$V_{i,j}^{t+1} = \frac{1}{2} (V_{i,j+1}^t + V_{i,j-1}^t) - \frac{\Delta x}{2 \Delta y} E_{i,j}^1 (V_{i,j+1}^t - V_{i,j-1}^t) - \Delta x E_{i,j}^2 V_{i+1,j}^t - \Delta x E_{i,j}^3$$

(2.6)

where again $V_{i,j}$, $E_{i,j}^1$, $E_{i,j}^2$, and $E_{i,j}^3$ are the approximated values of $v$, $E^1$, $E^2$, and $E^3$, respectively, at the point $(i \Delta x, j \Delta y)$, with $V_{0,j}$ the known initial conditions.

### 2.2 Reduced 2D Log-Elastographic Upwind and Central Difference Schemes

In this section, we establish a nonlinear reduced 2D Log-Elastographic scheme to compute solutions of first order partial differential equation systems, especially in cases when there is the potential for exponential growth. For example, in cases when the coefficient matrix, $E^1$, has real eigenvalues, the above system (2.4) has this potential when the real part of the eigenvalues of the coefficient matrix, $E^2$, have wrong signs. When solving these kinds of equations with standard linear methods, like the standard upwind scheme or central difference scheme mentioned
in the previous section, the numerical solution can diverge in time because of the 
exponential growth of the error in the solution. This exponential growth is caused 
by linear terms in the unknown solution together with the sign of the real part of 
the eigenvalues of the coefficient matrix, $E^2$, in this term. In order to control this 
kind of numerical blowup, one needs to set up a restriction on the spatial mesh size, 
which is not a CFL restriction on the time step. To remove this tiny spatial mesh 
size condition, a logarithmic representation was proposed in [21] for a problem which 
is different from our problem. The basic idea in [21] is to make a change of variables 
into new variables that scale logarithmically with the solution. The utilization of the 
logarithmic transformation enables removal of the instability due to the linear term 
that can cause numerical computations to blowup, or linear solvers not to converge. 
Inspired by [21] and the results in [32] for the acoustic equation, we apply a related 
idea to solve system (2.4).

To describe the method, we rewrite system (2.4) back in scalar form as

$$\tilde{\mu}_x + E^1(1,1)\tilde{\mu}_y + E^1(1,2)\tilde{\mu}_y + E^2(1,1)\tilde{\mu} + E^3(1) = 0 \quad (2.7)$$

$$\tilde{p}_x + E^1(2,1)\tilde{\mu}_y + E^1(2,2)\tilde{\mu}_y + E^2(2,1)\tilde{\mu} + E^3(2) = 0 \quad (2.8)$$

where $E^k(i,j)$ is the component of the matrix $E^k$ at the $i^{th}$ row and the $j^{th}$ column, 
k = 1, 2, and $E^3(j)$ is the $j^{th}$ component of the vector $E^3$. If we look at equations 
(2.7) and (2.8) carefully, we notice that both equations have terms which are linear 
in $\tilde{\mu}$ and could cause exponential growth of $\tilde{\mu}$ while neither equation has a linear 
term in $\tilde{p}$. To accurately compute, if we use the linear scheme of the previous 
section, we may need a very fine discretization, required by a spatial mesh size 
restriction similar to the one mentioned above. In order to compute with a coarser 
discretization, we take another approach.

To develop our new algorithm, which will be nonlinear, we divide equation 
(2.7) by $\tilde{\mu}$, a step also in [32], and introduce a new variable $v = \log \tilde{\mu}$, where we 
note that for our targeted $v$ the values are always positive since for our application 
requirement $\tilde{\mu} \geq 1m^2/s^2$. Then equations (2.7) and (2.8) become, in the variables
\[ v, \hat{\mu}, \hat{p}, \]
\[ v_x + E^1(1, 1)v_y + E^1(1, 2)\hat{p}_y e^{-v} + E^2(1, 1) + E^3(1)e^{-v} = 0 \quad (2.9) \]
\[ \hat{p}_x + E^1(2, 1)\mu v_y + E^1(2, 2)\hat{p}_y + E^2(2, 1)\hat{p} + E^3(2) = 0 \quad (2.10) \]

where in some places we have replaced \( \exp(v) \) by \( \tilde{\mu} \).

Now we have two equations: one is for \( v \) and the other is our previous equation for \( \hat{p} \). The first step is to discretize (2.9) and (2.10) with the standard upwind scheme or the stable central difference scheme given in Section 2.1.

Writing (2.9) and (2.10) in system form of \[ \begin{pmatrix} v \\ \hat{p} \end{pmatrix} = w, \]
\[ w_x + A\tilde{\mu}w_y + E\tilde{\mu} = 0 \quad (2.11) \]

where
\[ A\tilde{\mu} = \begin{pmatrix} \frac{\tilde{u}_{2,y}}{\tilde{u}_{2,x}} & \frac{1}{\tilde{u}_{2,x} \tilde{\mu}} \\ (\tilde{u}_{1,y} - \frac{\tilde{u}_{2,y}}{\tilde{u}_{2,x}} \tilde{u}_{1,x})\tilde{\mu} - \frac{\tilde{u}_{1,x}}{\tilde{u}_{2,x}} \end{pmatrix}, \]
\[ E\tilde{\mu} = \begin{pmatrix} \frac{\Delta \tilde{u}_{2}}{\tilde{u}_{2,x}} + \omega_c^2 \frac{\tilde{u}_{2}}{\tilde{u}_{2,x} \tilde{\mu}} \\ (\Delta \tilde{u}_1 - \frac{\Delta \tilde{u}_{2}}{\tilde{u}_{2,x}} \tilde{u}_{1,x})\tilde{\mu} + \omega_c^2 (\tilde{u}_1 - \frac{\tilde{u}_{2}}{\tilde{u}_{2,x}} \tilde{u}_{1,x}) \end{pmatrix} \]
both depend nonlinearly on \( \tilde{\mu} \).

Applying the standard upwind scheme to discretize (2.11), we would obtain
\[ W_{i+1,j} = W_{i,j} - \Delta x \left( A_{i,j}^+ (W_{i,j} - W_{i,j-1}) \right. \]
\[ \left. + A_{i,j}^- (W_{i,j+1} - W_{i,j}) \right) - \Delta x E_{i,j}^\tilde{\mu} \quad (2.12) \]

where \( W_{i,j} \) is the approximated values of \( w \), \( A_{i,j}^+, A_{i,j}^- \), \( E_{i,j}^\tilde{\mu} \) are the values of \( A\tilde{\mu}^+, E\tilde{\mu} \), respectively, at the point \((i\Delta x, j\Delta y)\), with \( W_{0,j} \) the known initial conditions.

This discretization appears to require that at each step we need to recalculate the eigenvalues and eigenvectors for \( A\tilde{\mu} \), which contains the computed value of \( \mu = \log v \).

To avoid this extra calculation which can lead to additional inaccuracies, we establish
relationships between the matrix $E^1$ in (2.4) and the matrix $A^\mu$ in (2.11) and between $E^{1,\pm}$ and $A^\mu_{\pm}$.

If we write

$$E^1 = \begin{pmatrix} \hat{u}_{2,y} & \frac{1}{\hat{u}_{2,x}} \\ \hat{u}_{1,y} - \frac{\hat{u}_{2,y}}{\hat{u}_{2,x}} \hat{u}_{1,x} & -\frac{\hat{u}_{1,x}}{\hat{u}_{2,x}} \end{pmatrix} \text{ as } \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in its undiscretized form and assume the eigenvalues of $E^1$ are $\lambda_1$ and $\lambda_2$, where $\lambda_1$ denotes the eigenvalue with positive real part and $\lambda_2$ denotes the eigenvalue with negative real part when they have opposite signs, then it is straightforward that $A^\mu$, the eigenvector matrix $S$ of $E^1$ together with its inverse matrix $S^{-1}$ can be expressed as

$$A^\mu = \begin{pmatrix} a & b/\mu \\ c/\mu & d \end{pmatrix}, S = \begin{pmatrix} b & \lambda_2 - d \\ \lambda_1 - a & c \end{pmatrix}, S^{-1} = \frac{1}{\alpha} \begin{pmatrix} c & d - \lambda_2 \\ a - \lambda_1 & b \end{pmatrix},$$

where $\alpha = bc - \lambda_1 \lambda_2 + a \lambda_2 + d \lambda_1 - ad$. And hence

$$E^{1,+} = SA^+ S^{-1} = \frac{1}{\alpha} \begin{pmatrix} bc \lambda_1 & b \lambda_1 (d - \lambda_2) \\ \lambda_1 (\lambda_1 - a) c & \lambda_1 (\lambda_1 - a) (d - \lambda_2) \end{pmatrix},$$

$$E^{1,-} = SA^- S^{-1} = \frac{1}{\alpha} \begin{pmatrix} \lambda_2 (\lambda_2 - d) (a - \lambda_1) & b \lambda_2 (\lambda_2 - d) \\ \lambda_2 (a - \lambda_1) c & bc \lambda_2 \end{pmatrix}. \tag{2.13}$$

Note that when the real parts of the eigenvalues both have positive (negative) real parts, then $E^{1,-}$ (or $E^{1,+}$) are zero and $E^{1,+} = E^1$ (or $E^{1,-} = E^1$). A straightforward calculation yields that $E^1$ and $A^\mu$ have the same eigenvalues. And hence the eigenvector matrix $S^\mu$ of $A^\mu$ and its inverse $(S^\mu)^{-1}$ are

$$S^\mu = \begin{pmatrix} b/\mu & \lambda_2 - d \\ \lambda_1 - a & c/\mu \end{pmatrix}, (S^\mu)^{-1} = \frac{1}{\alpha} \begin{pmatrix} c/\mu & d - \lambda_2 \\ a - \lambda_1 & b/\mu \end{pmatrix}. $$
So we can write

\[ A^\mu_+ = S^\mu \Lambda^+(S^\mu)^{-1} = \frac{1}{\alpha} \begin{pmatrix} bc\lambda_1 & \frac{b}{\mu_\lambda_1}(d - \lambda_2) \\ \lambda_1(\lambda_1 - a)c\tilde{\mu} & \lambda_1(\lambda_1 - a)(d - \lambda_2) \end{pmatrix}, \]

\[ A^\mu_- = S^\mu \Lambda^-(S^\mu)^{-1} = \frac{1}{\alpha} \begin{pmatrix} \lambda_2(\lambda_2 - d)(a - \lambda_1) & \frac{b}{\mu_\lambda_2}(d - \lambda_2) \\ \lambda_2(a - \lambda_1)c\tilde{\mu} & bc\lambda_2 \end{pmatrix}. \]

Therefore, the following relationships between the components of \( E^{1, \pm} \) and the components of \( A^\mu_\pm \) hold:

\[ A^\mu_\pm(1, 1) = E^{1, \pm}(1, 1), \quad A^\mu_\pm(1, 2) = E^{1, \pm}(1, 2)/\tilde{\mu}, \]
\[ A^\mu_\pm(2, 1) = E^{1, \pm}(2, 1)/\tilde{\mu}, \quad A^\mu_\pm(2, 2) = E^{1, \pm}(2, 2). \]

Because these relationships are either linear in \( \tilde{\mu} \) or \( 1/\tilde{\mu} \), we can then write the standard upwind scheme for (2.11) as follows

\[ \tilde{v}_{i+1,j} = \tilde{v}_{i,j} - \frac{\Delta x}{\Delta y} \left( E^{1,+}_{i,j}(1, 1)(\tilde{v}_{i,j} - \tilde{v}_{i,j-1}) + E^{1,+}_{i,j}(1, 2)(\tilde{p}_{i,j} - \tilde{p}_{i,j-1})/\tilde{\mu}_{i,j} \right. \]
\[ + E^{1,-}_{i,j}(1, 1)(\tilde{v}_{i,j+1} - \tilde{v}_{i,j}) + E^{1,-}_{i,j}(1, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j})/\tilde{\mu}_{i,j} \]
\[ - \Delta x E^{\tilde{\mu}}_{i,j}(1) \]

\[ \tilde{p}_{i+1,j} = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( E^{1,+}_{i,j}(2, 1)(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + E^{1,+}_{i,j}(2, 2)(\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) \right. \]
\[ + E^{1,-}_{i,j}(2, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j}) + E^{1,-}_{i,j}(2, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \]
\[ - \Delta x E^{\tilde{\mu}}_{i,j}(2) \] (2.15)

where \( \tilde{v}, \tilde{\mu} \) and \( \tilde{p} \) are the approximate values of \( v, \mu \) and \( p \) determined by the discretized system.

The second step is to take the exponential of the first discretized equation.
Then we obtain

\[
\begin{align*}
\tilde{\mu}_{i+1,j} & = \frac{1}{\mu_{i,j}} \left( 1 - \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1) \right) \right) \cdot \frac{\Delta x}{\Delta y} E_{i,j}^{1,+}(1,1) \cdot \frac{\Delta x}{\Delta y} E_{i,j}^{1,-}(1,1) \\
& \quad \cdot \exp \left( - \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(1,2) (\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) + E_{i,j}^{1,-}(1,2) (\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \right) / \tilde{\mu}_{i,j} \\
& \quad - \Delta x \left( E_{i,j}^{2}(1,1) + E_{i,j}^{3}(1) / \tilde{\mu}_{i,j} \right) \right) \\
\tilde{p}_{i+1,j} & = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(2,1) (\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + E_{i,j}^{1,-}(2,1) (\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j}) \\
& \quad + E_{i,j}^{1,+}(2,2) (\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) + E_{i,j}^{1,-}(2,2) (\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \right) \\
& \quad - \Delta x \left( E_{i,j}^{2}(2,1) \tilde{\mu}_{i,j} + E_{i,j}^{3}(2) \right) \tag{2.17}
\end{align*}
\]

where we replace \( E_{i,j}^{\tilde{\mu}}(1) \) by \( E_{i,j}^{2}(1,1) + E_{i,j}^{3}(1) / \tilde{\mu}_{i,j} \) in the first equation and \( E_{i,j}^{\tilde{\mu}}(2) \) by \( E_{i,j}^{2}(2,1) \tilde{\mu}_{i,j} + E_{i,j}^{3}(2) \) in the second equation.

Now we apply the approximation \( \exp (b \Delta x) \approx 1 + b \Delta x \) to the exponential term that contains the pressure in the first equation, then we get

\[
\begin{align*}
\tilde{\mu}_{i+1,j} & = \left( \tilde{\mu}_{i,j} \left( 1 - \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1) \right) \right) \right) \cdot \frac{\Delta x}{\Delta y} E_{i,j}^{1,+}(1,1) \cdot \frac{\Delta x}{\Delta y} E_{i,j}^{1,-}(1,1) \\
& \quad - \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(1,2) (\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + E_{i,j}^{1,-}(1,2) (\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j}) \right) \\
& \quad \cdot \exp \left( - \Delta x \left( E_{i,j}^{2}(1,1) + E_{i,j}^{3}(1) / \tilde{\mu}_{i,j} \right) \right) \tag{2.18}
\end{align*}
\]

Note that we did not approximate the exponential term

\[
\exp \left( - \Delta x \left( E_{i,j}^{2}(1,1) + E_{i,j}^{3}(1) / \tilde{\mu}_{i,j} \right) \right).
\]

The reason is that the term in the exponent is often near zero especially in the regions where the scaled shear modulus is almost constant, see [31] for the proof that this is also the case for the acoustic wave equation. We solve the discretized equation (2.17) first and then (2.18) since one is linear and the other one is nonlinear. We call this method the reduced 2D Log-Elastographic upwind method for the reduced elastic system.
Similarly, if we discretize equations (2.9) and (2.10) with the stable central difference scheme, we obtain

\[
\tilde{v}_{i+1,j} = \frac{1}{2}(\tilde{v}_{i,j+1} + \tilde{v}_{i,j-1}) - \frac{\Delta x}{2} \left( E_{i,j}^1(1,1)(\tilde{v}_{i,j+1} - \tilde{v}_{i,j-1}) + E_{i,j}^1(1,2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1})/\tilde{\mu}_{i,j} \right) - \Delta x \left( E_{i,j}^2(1,1) + E_{i,j}^3(1)/\tilde{\mu}_{i,j} \right) \tag{2.19}
\]

\[
\tilde{p}_{i+1,j} = \frac{1}{2}(\tilde{p}_{i,j+1} + \tilde{p}_{i,j-1}) - \frac{\Delta x}{2} \left( E_{i,j}^3(2,1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) + E_{i,j}^3(2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1}) \right) - \Delta x \left( E_{i,j}^2(2,1)\tilde{\mu}_{i,j} + E_{i,j}^3(2) \right). \tag{2.20}
\]

Again we can take the exponential of the first equation and approximate \( \exp(b\Delta x) \), where \( b \) contains the discretized pressure, by \( 1 + b\Delta x \) to obtain

\[
\tilde{\mu}_{i+1,j} = \left( \frac{\Delta x}{2} \left( E_{i,j}^1(1,1) \right) (\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) - \frac{\Delta x}{2} \left( E_{i,j}^3(2,1) \tilde{\mu}_{i,j} + E_{i,j}^3(2) \right) \right) \cdot \exp \left( -\Delta x \left( E_{i,j}^2(1,1) + E_{i,j}^3(1)/\tilde{\mu}_{i,j} \right) \right). \tag{2.21}
\]

We call this scheme the reduced 2D Log-Elastographic central difference algorithm for the reduced elastic system.
CHAPTER 3
Inversion Algorithm Using the Elastic System

In this chapter, we consider the following full 2D plane strain elastic system

\[ u_{tt} = \nabla \cdot (\tilde{\mu}(\nabla u + (\nabla u)^T)) + \nabla \tilde{p} + \tilde{f} \]

\[ = \tilde{\mu} \Delta u + \nabla \tilde{\mu} \cdot (\nabla u + (\nabla u)^T) + (1 + \epsilon \tilde{\mu}) \nabla \tilde{p} + \tilde{f}. \tag{3.1} \]

Note that we defined \((\lambda/\rho) \nabla \cdot u = \tilde{p}\) in Chapter 1, and hence \(\nabla \cdot (\nabla u^T) = \epsilon \nabla \tilde{p}\) where \(\epsilon = \rho/\lambda\) and it is assumed that \(\rho\) and \(\lambda\) are both constants. As in the reduced equation case, our algorithm will still be focused on the frequency domain case:

\[ \tilde{\mu} \Delta \hat{u} + \nabla \tilde{\mu} \cdot (\nabla \hat{u} + (\nabla \hat{u})^T) + (1 + \epsilon \tilde{\mu}) \nabla \tilde{p} + \omega_c^2 \hat{u} = 0. \tag{3.2} \]

Our object is to solve the above Fourier transformed equation (3.2) for \(\tilde{\mu}\) and \(\hat{p}\) with \(\hat{u}\) known. But in the full 2D plane strain elastic system case, there are more terms: one of them includes \(\epsilon \tilde{\mu} \nabla \tilde{p}\), which causes our problem to be nonlinear in \(\tilde{\mu}\) and \(\hat{p}\). Theoretically, this term is very small, because \(\epsilon \nabla \tilde{p} = \nabla \cdot (\nabla u^T) = \nabla (\nabla \cdot u)\) and \(\nabla \cdot u\) is very small. We handle this in two ways: one is to neglect the nonlinear term and solve the remaining system; the other one is to include this nonlinear term and use an alternating method to make our computations. For this alternating method, we use \(\tilde{\mu}\) obtained from the previous step when computing the term \(\tilde{\mu} \nabla \tilde{p}\) at the present step and when we are solving for \(\tilde{\mu}\) at the present step. Then we use the \(\tilde{\mu}\) obtained at the present step when computing the term \(\tilde{\mu} \nabla \tilde{p}\) and when also we are solving for \(\hat{p}\) at the present step. We will compare the recoveries obtained from these two different ways in Chapter 5 and show that the differences between the recoveries are very small.
3.1 Inversion Algorithm when the Nonlinear Term is Neglected

If we neglect the nonlinear term \( \epsilon \tilde{\mu} \nabla \tilde{p} \) in system (3.2) we obtain

\[
\tilde{\mu} \Delta \tilde{u} + \nabla \tilde{\mu} \cdot (\nabla \tilde{u} + (\nabla \tilde{u})^T) + \nabla \tilde{p} + \omega_c^2 \tilde{u} = 0.
\] (3.3)

Rewriting the above first order system (3.3), we obtain the following equation system written in matrix form

\[
\begin{pmatrix}
2 \hat{u}_{1,x} & 1 \\
\hat{u}_{2,x} + \hat{u}_{1,y} & 0
\end{pmatrix} \begin{pmatrix}
\tilde{\mu}_x \\
\tilde{p}_x
\end{pmatrix} + \begin{pmatrix}
\hat{u}_{1,y} + \hat{u}_{2,x} & 0 \\
2 \hat{u}_{2,y} & 1
\end{pmatrix} \begin{pmatrix}
\tilde{\mu}_y \\
\tilde{p}_y
\end{pmatrix} + \begin{pmatrix}
\Delta \hat{u}_1 & 0 \\
\Delta \hat{u}_2 & 0
\end{pmatrix} \begin{pmatrix}
\hat{\mu} \\
\hat{p}
\end{pmatrix} + \omega_c^2 \begin{pmatrix}
\hat{u}_1 \\
\hat{u}_2
\end{pmatrix} = 0.
\] (3.4)

Denoting

\[
v_f := \begin{pmatrix}
\tilde{\mu} \\
\tilde{p}
\end{pmatrix}, \quad A_f := \begin{pmatrix}
2 \hat{u}_{1,x} & 1 \\
\hat{u}_{2,x} + \hat{u}_{1,y} & 0
\end{pmatrix}, \quad B_f := \begin{pmatrix}
\hat{u}_{1,y} + \hat{u}_{2,x} & 0 \\
2 \hat{u}_{2,y} & 1
\end{pmatrix}, \quad \text{and} \quad C_f := \begin{pmatrix}
\Delta \hat{u}_1 & 0 \\
\Delta \hat{u}_2 & 0
\end{pmatrix},
\]

then \( A_f^{-1} = \begin{pmatrix}
0 & 1 \\
1 & \frac{1}{\hat{u}_{2,x} + \hat{u}_{1,y}}
\end{pmatrix} \). Multiplying \( A_f^{-1} \) on both sides of (3.4), we obtain another system of first order equations of the form

\[
v_{f,x} + \mathcal{F}_1 v_{f,y} + \mathcal{F}_2^2 v_f + \mathcal{F}_3^3 = 0
\] (3.5)

where

\[
\mathcal{F}_1 = \begin{pmatrix}
\frac{2 \hat{u}_{2,y}}{\hat{u}_{2,x} + \hat{u}_{1,y}} & \frac{1}{\hat{u}_{2,x} + \hat{u}_{1,y}} \\
\hat{u}_{2,x} + \hat{u}_{1,y} & -\frac{2 \hat{u}_{1,x}}{\hat{u}_{2,x} + \hat{u}_{1,y}}
\end{pmatrix},
\]

\[
\mathcal{F}_2^2 = \begin{pmatrix}
\frac{\Delta \hat{u}_2}{\hat{u}_{2,x} + \hat{u}_{1,y}} & 0 \\
\Delta \hat{u}_1 - \frac{2 \hat{u}_{1,x}}{\hat{u}_{2,x} + \hat{u}_{1,y}} \Delta \hat{u}_2 & 0
\end{pmatrix}, \quad \text{and} \quad \mathcal{F}_3^3 = \begin{pmatrix}
\frac{\omega_c^2 \hat{u}_2}{\hat{u}_{2,x} + \hat{u}_{1,y}} \\
\omega_c^2 \hat{u}_1 - \frac{2 \omega_c^2 \hat{u}_{1,x}}{\hat{u}_{2,x} + \hat{u}_{1,y}}
\end{pmatrix}.
Here, to obtain the linear system (3.5), we have assumed that \( \hat{u}_{2,x} + \hat{u}_{1,y} \neq 0 \). With this assumption, we can interpret as before that the \( x \) direction is a time-like direction. In keeping with this interpretation we assume that \( v_f(0,y) \) is known. We discretize the system (3.5) with the following schemes introduced in Sections 2.1, 2.2, and solve the discretized system (or equations) respectively as described in Sections 2.1, 2.2.

(I) The Partially Implicit Upwind Scheme when the Term, \( \epsilon \tilde{\mu} \nabla \tilde{p} \), is Neglected:

\[
V_{i+1,j} = V_{i,j} - \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(V_{i,j} - V_{i,j-1}) + F_{i,j}^{1,-}(V_{i,j+1} - V_{i,j}) \right)
- \Delta x F_{i,j}^2 V_{i+1,j} - \Delta x F_{i,j}^3
\]  

(3.6)

where \( V_{i,j} \) is the approximated values of \( v_f \), \( F_{i,j}^{1,+} \), \( F_{i,j}^2 \), and \( F_{i,j}^3 \) are the evaluated values of \( F^{1,+} \), \( F^2 \), and \( F^3 \), respectively, at the point \((i\Delta x, j\Delta y)\), with \( V_0 \) the known initial conditions. \( F^{1,\pm} = \Lambda^{\pm} S^{-1} \), and \( \Lambda^+ (\Lambda^-) \) has all those nonnegative (negative) eigenvalues, under the assumption that the eigenvalues are real, of \( F^1 \) on its diagonal, \( S \) is the eigenvector matrix of \( F^1 \).

(II) The Stable Central Difference Scheme when the Term, \( \epsilon \tilde{\mu} \nabla \tilde{p} \), is Neglected:

\[
V_{i+1,j} = \frac{1}{2}(V_{i,j+1} + V_{i,j-1}) - \frac{\Delta x}{2\Delta y} F_{i,j}^1 (V_{i,j+1} - V_{i,j-1})
- \Delta x F_{i,j}^2 V_{i+1,j} - \Delta x F_{i,j}^3.
\]  

(3.7)

(III) The 2D Log-Elastographic Nonlinear Upwind Scheme when the Term, \( \epsilon \tilde{\mu} \nabla \tilde{p} \), is Neglected:

\[
\tilde{\mu}_{i+1,j} = \left( \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(1,1) - F_{i,j}^{1,-}(1,1) \right) \cdot \frac{\Delta x}{\Delta y} F_{i,j}^{1,+}(1,1) \cdot \frac{\Delta x}{\Delta y} F_{i,j}^{1,-}(1,1) \right)
- \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(1,2) (\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + F_{i,j}^{1,-}(1,2) (\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j}) \right)
\cdot \exp \left( - \Delta x (F_{i,j}^2(1,1) + F_{i,j}^3(1)/\tilde{\mu}_{i,j}) \right)
\]  

(3.8)
\[
\tilde{p}_{i+1,j} = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+} \hat{\mu}_{i,j} - F_{i-1,j}^{1,-} - F_{i,j+1}^{1,-}(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + F_{i,j}^{1,+}(\hat{\mu}_{i,j} - \hat{\mu}_{i,j-1}) \right) \\
+ F_{i,j}^{1,+}(2,2)(\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) + F_{i,j}^{1,+}(2,2)(\hat{p}_{i,j+1} - \hat{p}_{i,j}) \\
- \Delta x \left( F_{i,j}^{2}(2,1)\tilde{\mu}_{i,j} + F_{i,j}^{3}(2) \right)
\]

(3.9)

where \(\tilde{\mu}_{i,j}, \tilde{p}_{i,j}\) are the approximated values of \(\hat{\mu}, \hat{p}\), \(\hat{F}_{i,j}^{1,+}(m,n), \hat{F}_{i,j}^{2}(m,n),\) and \(\hat{F}_{i,j}^{3}(n)\) are the evaluated values of \(F^{1,+}(m,n), F^{2}(m,n),\) and \(F^{3}(n), m, n = 1, 2,\) respectively, at the point \((i \Delta x, j \Delta y)\), with \(\tilde{\mu}_{0,j}\) and \(\tilde{p}_{0,j}\) the known initial conditions.

(IV) The 2D Log-Elastographic Nonlinear Central Difference Scheme when the Term, \(\epsilon \hat{\mu} \nabla \hat{p}\), is Neglected:

\[
\tilde{\mu}_{i+1,j} = \left( \frac{\hat{\mu}_{i+1,j}}{\hat{\mu}_{i-1,j}}, \frac{\hat{\mu}_{i,j+1}}{\hat{\mu}_{i,j-1}} \right) \frac{\Delta x}{2 \Delta y} \left( F_{i,j}^{2}(1,2) \tilde{p}_{i,j+1} + F_{i,j}^{3}(1) \right) \\
\cdot \exp \left( - \Delta x \left( F_{i,j}^{2}(1,1) + F_{i,j}^{3}(1) / \hat{\mu}_{i+1,j} \right) \right)
\]

(3.10)

\[
\tilde{p}_{i+1,j} = \frac{1}{2} (\tilde{p}_{i,j+1} + \tilde{p}_{i,j-1}) - \frac{\Delta x}{2 \Delta y} \left( F_{i,j}^{2}(2,1) \tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1} \right) \\
+ F_{i,j}^{1}(2,2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1}) - \Delta x \left( F_{i,j}^{2}(2,1) \tilde{\mu}_{i,j} + F_{i,j}^{3}(2) \right).
\]

(3.11)

3.2 Inversion Algorithm when the Nonlinear Term is Included

Now we consider the original full elastic system:

\[
\hat{\mu} \Delta \hat{u} + \nabla \hat{\mu} \cdot (\nabla \hat{u} + (\nabla \hat{u})^T) + (1 + \epsilon \hat{\mu}) \nabla \hat{p} + \omega^2 \hat{u} = 0.
\]

(3.12)

Instead of neglecting the term \(\epsilon \hat{\mu} \nabla \hat{p}\), we keep it and regard \(\epsilon \hat{\mu}\) as a coefficient and \(\hat{p}\) as unknown in this term. We solve these equations for \(\tilde{\mu}, \tilde{p}\) using an alternating method: solve first for \(\tilde{\mu}\) and then for \(\tilde{p}\). Since we regard the \(\epsilon \hat{\mu}\) term as a coefficient, when we solve for \(\tilde{p}\) we use the value of \(\tilde{\mu}\) calculated in the immediately preceding
step. Now the Fourier transformed system (3.12) written in matrix form is as follows:

\[
\begin{pmatrix}
2\hat{u}_{1,x} & 1 + \epsilon\hat{\mu} \\
\hat{u}_{2,x} + \hat{u}_{1,y} & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mu}_x \\
\hat{p}_x
\end{pmatrix}
+ \begin{pmatrix}
\hat{u}_{1,y} + \hat{u}_{2,x} & 0 \\
2\hat{u}_{2,y} & 1 + \epsilon\hat{\mu}
\end{pmatrix}
\begin{pmatrix}
\hat{\mu}_y \\
\hat{p}_y
\end{pmatrix}
+ \begin{pmatrix}
\Delta\hat{u}_1 & 0 \\
\Delta\hat{u}_2 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{u}_1 \\
\hat{u}_2
\end{pmatrix}
= 0. \tag{3.13}
\]

Let

\[
\bar{A}_f := \begin{pmatrix}
2\hat{u}_{1,x} & 1 + \epsilon\hat{\mu} \\
\hat{u}_{2,x} + \hat{u}_{1,y} & 0
\end{pmatrix}, \quad \bar{B}_f := \begin{pmatrix}
\hat{u}_{1,y} + \hat{u}_{2,x} & 0 \\
2\hat{u}_{2,y} & 1 + \epsilon\hat{\mu}
\end{pmatrix},
\]

and assume \(1 + \epsilon\tilde{\mu} \neq 0\) and as before that \(\hat{u}_{2,x} + \hat{u}_{1,y} \neq 0\), then

\[
\bar{A}_f^{-1} = \begin{pmatrix}
0 & -\frac{1}{\bar{u}_{2,x} + \bar{u}_{1,y}} \\
\frac{1}{1 + \epsilon\hat{\mu}} & -\frac{2\hat{u}_{1,x}}{(1 + \epsilon\hat{\mu})(\bar{u}_{2,x} + \bar{u}_{1,y})}
\end{pmatrix}.
\]

Multiplying \(\bar{A}_f^{-1}\) on both sides of (3.13), we arrive at the following system of first order equations

\[
v_{f,x} + F^1 v_{f,y} + F^2 v_f + F^3 = 0 \tag{3.14}
\]

where

\[
\tilde{F}^1 = \begin{pmatrix}
\frac{2\hat{u}_{2,y}}{\bar{u}_{2,x} + \bar{u}_{1,y}} - \frac{4\hat{u}_{1,x}\hat{u}_{2,x}}{(\bar{u}_{2,x} + \bar{u}_{1,y})(1 + \epsilon\hat{\mu})} - \frac{1 + \epsilon\hat{\mu}}{\bar{u}_{2,x} + \bar{u}_{1,y}} \\
F^1(1, 1) & F^1(1, 2)(1 + \epsilon\tilde{\mu}) \\
F^1(2, 1)/(1 + \epsilon\tilde{\mu}) & F^1(2, 2)
\end{pmatrix}, \tag{3.15}
\]

\[
\tilde{F}^2 = \begin{pmatrix}
\frac{\Delta\hat{u}_1}{(1 + \epsilon\hat{\mu})} - \frac{2\hat{u}_{1,x}}{(\bar{u}_{2,x} + \bar{u}_{1,y})(1 + \epsilon\hat{\mu})}\Delta\hat{u}_2 & 0 \\
\omega^2\hat{u}_1 & \frac{2\omega^2\hat{u}_{1,x}}{(\bar{u}_{2,x} + \bar{u}_{1,y})(1 + \epsilon\hat{\mu})}
\end{pmatrix}, \text{ and}
\]

\[
\tilde{F}^3 = \begin{pmatrix}
\frac{\omega^2\hat{u}_1}{(1 + \epsilon\hat{\mu})} - \frac{2\omega^2\hat{u}_{1,x}}{(\bar{u}_{2,x} + \bar{u}_{1,y})(1 + \epsilon\hat{\mu})}
\end{pmatrix}.
\]
Note that $\epsilon \mu$ is now contained in the coefficient matrices $\bar{F}^j$, $j = 1, 2, 3$.

We discretize the above system (3.14) with the standard partially implicit upwind scheme, the stable partially implicit central difference scheme, the 2D Log-Elastographic nonlinear upwind and central difference schemes respectively.

(I) The Standard Partially Implicit Upwind Scheme when the Nonlinear Term, $\epsilon \mu \nabla \hat{\mu}$, is Included:

$$
V_{i+1,j} = V_{i,j} - \frac{\Delta x}{\Delta y} \left( \bar{F}^{1,+}_{i,j} (V_{i,j} - V_{i,j-1}) + \bar{F}^{1,-}_{i,j} (V_{i,j+1} - V_{i,j}) \right) \\
- \Delta x \bar{F}^2_{i,j} V_{i+1,j} - \Delta x \bar{F}^3_{i,j}.
$$

(3.16)

where $\bar{F}^{1,\pm} = S \tilde{\Lambda}^{\pm} S^{-1}$, and $\tilde{\Lambda}^{+} (\tilde{\Lambda}^-)$ has all those nonnegative (negative) eigenvalues, under the assumption that the eigenvalues are real, of $F^1$ on its diagonal, $S$ is the eigenvector matrix of $\bar{F}^1$. As usual, in order to apply the above standard upwind scheme (3.16), we need to calculate the eigenvalues and eigenvectors of the matrix $F^1$ to get $F^{1,+}$ and $F^{1,-}$. But here, because $\epsilon \mu$ appears in the matrix $F^1$, we need to calculate the eigenvalues and eigenvectors of $\bar{F}^1$ with the computed value of $\hat{\mu}$ at each step. As we mentioned before, this extra calculation can lead to additional inaccuracies. However, looking at the structure of the matrix $\bar{F}^1$ carefully, we can find relationships between matrices $\bar{F}^1$ and $F^1$. We follow the idea introduced in Section 2.2 where we found relationships between matrices $E^1$ and $\tilde{A} \tilde{\mu}$. Here similarly we establish that the eigenvalues of $\bar{F}^1$ and $F^1$ are the same and the components of $\bar{F}^1$ and $F^1$ satisfy the simple relationships

$$
\bar{F}^{1,\pm}(1,1) = F^{1,\pm}(1,1), \quad \bar{F}^{1,\pm}(1,2) = F^{1,\pm}(1,2)(1 + \epsilon \mu), \\
\bar{F}^{1,\pm}(2,1) = F^{1,\pm}(2,1)/(1 + \epsilon \mu), \quad \bar{F}^{1,\pm}(2,2) = F^{1,\pm}(2,2).
$$

This is not the same as the relationship between $E^1$ and $\tilde{A} \tilde{\mu}$, but it is similarly useful in the sense that the eigenvalue, eigenvector matrix calculations do not need to be made using the computed value of $\hat{\mu}$. Therefore, the standard partially implicit
upwind scheme (3.16) for system (3.14) can be rewritten in the form

\[
\tilde{\mu}_{i+1,j} = \left( \tilde{\mu}_{i,j} - \frac{\Delta x}{\Delta y} \left( F^1_{i,j+1}(1, 1)(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + F^1_{i,j+1}(1, 2)(1 + \epsilon \tilde{\mu}_{i,j})(\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) \\
+ F^1_{i,j-1}(1, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j}) + F^1_{i,j-1}(1, 2)(1 + \epsilon \tilde{\mu}_{i,j})(\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \right) - \Delta x F^3_{i,j}(1) \right) / (1 + \Delta x F^2_{i,j}(1, 1))
\]

(3.17)

\[
\tilde{p}_{i+1,j} = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( F^1_{i,j+1}(2, 1)(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1})/(1 + \epsilon \tilde{\mu}_{i,j+1,j}) + F^1_{i,j+1}(2, 2)(\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) \\
+ F^1_{i,j-1}(2, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j})/(1 + \epsilon \tilde{\mu}_{i+1,j}) + F^1_{i,j-1}(2, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \right) - \Delta x F^2_{i,j}(2, 1)\tilde{\mu}_{i+1,j} - \Delta x F^3_{i,j}(2).
\]

(3.18)

We solve the above two discretized equations for \(\tilde{\mu}\) and \(\tilde{p}\) with an alternating method: first we solve equation (3.17) for \(\tilde{\mu}\) at the \((i + 1)\)st step with the calculated value of \(\tilde{\mu}\) and \(\tilde{p}\) at the \(i\)th step; then we solve equation (3.18) for \(\tilde{p}\) at the \((i + 1)\)st step, with values of \(\tilde{p}\) at the \(i\)th step, and with the calculated value of \(\tilde{\mu}\) at the \(i\)th step for all except for the \(\epsilon \tilde{\mu}\) term where we use the value of \(\tilde{\mu}\) at the \((i + 1)\)st step.

(II) The Stable Central Difference Scheme when the Nonlinear Term, \(\epsilon \tilde{\mu} \nabla \tilde{p}\), is Included:

Similarly, applying the central difference method, we solve the following two discretized equations for \(\tilde{\mu}\) and \(\tilde{p}\) in such a way that at the \((i + 1)\)st step we solve the first equation below for \(\tilde{\mu}_{i+1,j}\) and then the second equation for \(\tilde{p}_{i+1,j}\) with \(\tilde{\mu}_{i+1,j}\) in the term \(1 + \epsilon \tilde{\mu}\).

\[
\tilde{\mu}_{i+1,j} = \left( \frac{1}{2}(\tilde{\mu}_{i,j+1} + \tilde{\mu}_{i,j-1}) - \frac{\Delta x}{2\Delta y} \left( F^1_{i,j}(1, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) \\
+ F^1_{i,j}(1, 2)(1 + \epsilon \tilde{\mu}_{i,j})(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1}) \right) - \Delta x F^3_{i,j}(1) \right) / (1 + \Delta x F^2_{i,j}(1, 1))
\]

\[
\tilde{p}_{i+1,j} = \frac{1}{2}(\tilde{p}_{i,j+1} + \tilde{p}_{i,j-1}) - \frac{\Delta x}{2\Delta y} \left( F^1_{i,j}(2, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1})/(1 + \epsilon \tilde{\mu}_{i+1,j}) \\
+ F^1_{i,j}(2, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1}) \right) - \Delta x F^2_{i,j}(2, 1)\tilde{\mu}_{i+1,j} - \Delta x F^3_{i,j}(2).
\]

(III) The 2D Log-Elastographic Upwind Scheme when the Nonlinear Term, \(\epsilon \tilde{\mu} \nabla \tilde{p}\), is Included:
As before, we rewrite system (3.14) back in scalar form as
\begin{align*}
\tilde{\mu}_x + \tilde{F}^1(1,1)\tilde{\mu}_y + \tilde{F}^1(1,2)\tilde{p}_y + \tilde{F}^2(1,1)\tilde{\mu} + \tilde{F}^3(1) &= 0 \quad (3.19) \\
\tilde{p}_x + \tilde{F}^1(2,1)\tilde{\mu}_y + \tilde{F}^1(2,2)\tilde{p}_y + \tilde{F}^2(2,1)\tilde{\mu} + \tilde{F}^3(2) &= 0. \quad (3.20)
\end{align*}

Divide equation (3.19) by \( \tilde{\mu} \) and let \( v = \log \tilde{\mu} \). Then equations (3.19) and (3.20) become, in the variables \( v, \tilde{p}, \)
\begin{align*}
v_x + \tilde{F}^1(1,1)v_y + \tilde{F}^1(1,2)\tilde{p}_y/\tilde{\mu} + \tilde{F}^2(1,1) + \tilde{F}^3(1)/\tilde{\mu} &= 0 \quad (3.21) \\
\tilde{p}_x + \tilde{F}^1(2,1)\tilde{\mu}v_y + \tilde{F}^1(2,2)\tilde{p}_y + \tilde{F}^2(2,1)\tilde{\mu} + \tilde{F}^3(2) &= 0 \quad (3.22)
\end{align*}

where in some places we have replaced \( \exp(v) \) by \( \tilde{\mu} \). As before, the first step is to discretize (3.21) and (3.22) with the standard upwind scheme. Writing (3.21) and (3.22) in system form \( \begin{pmatrix} v \\ \tilde{p} \end{pmatrix} = w_f \), we obtain
\begin{align*}
w_{f,x} + \tilde{F}_1 w_{f,y} + \tilde{F}_2 &= 0 \quad (3.23)
\end{align*}

where
\begin{align*}
\tilde{F}_1 &= \begin{pmatrix}
\tilde{F}^1(1,1) & \tilde{F}^1(1,2)/\tilde{\mu} \\
\tilde{F}^1(2,1)\tilde{\mu} & \tilde{F}^1(2,2)
\end{pmatrix}, \\
\tilde{F}_2 &= \begin{pmatrix}
\Delta \tilde{u}_2 \mu (1+\epsilon\mu) - \frac{\Delta \tilde{u}_2}{(\tilde{u}_{2,x}+\tilde{u}_{1,y})(1+\epsilon\mu)} & \frac{\omega_2^2 \tilde{u}_2}{(\tilde{u}_{2,x}+\tilde{u}_{1,y})(1+\epsilon\mu)} \\
\frac{\Delta \tilde{u}_2}{(\tilde{u}_{2,x}+\tilde{u}_{1,y})(1+\epsilon\mu)} & \Delta \tilde{u}_2 + \frac{\omega_2^2 \tilde{u}_2}{(\tilde{u}_{2,x}+\tilde{u}_{1,y})(1+\epsilon\mu)} - \frac{2\omega_2^2 \tilde{u}_2 \tilde{u}_{1,x}}{(\tilde{u}_{2,x}+\tilde{u}_{1,y})(1+\epsilon\mu)}
\end{pmatrix}, \quad (3.24)
\end{align*}

where both depend nonlinearly on \( \tilde{\mu} \). Note also that using the standard upwind
scheme, we would obtain

\[
W_{i+1,j} = W_{i,j} - \frac{\Delta x}{\Delta y} \left( \tilde{F}_{i,j}^{1+} (W_{i,j} - W_{i,j-1}) + \tilde{F}_{i,j}^{1+} (W_{i,j+1} - W_{i,j}) \right) - \Delta x \tilde{F}_{i,j}^{2}.
\]  

(3.25)

Since \( \bar{F}^{1} = \begin{pmatrix} \bar{F}^{1}(1,1) & \bar{F}^{1}(1,2) / \bar{\mu} \\ \bar{F}^{1}(2,1) \bar{\mu} & \bar{F}^{1}(2,2) \end{pmatrix} \), \( \bar{F}^{1} \) and \( \bar{F}^{1} \) have the same eigenvalues and

the eigenvector matrix \( \bar{S} \) of \( \bar{F}^{1} \) satisfies

\[
\bar{S} = \begin{pmatrix} \bar{S}(1,1) / \bar{\mu} & \bar{S}(1,2) \\ S(2,1) & S(2,2) \bar{\mu} \end{pmatrix}, \quad \bar{S}^{-1} = \begin{pmatrix} \bar{S}^{-1}(1,1) \bar{\mu} & \bar{S}^{-1}(1,2) \\ \bar{S}^{-1}(2,1) & \bar{S}^{-1}(2,2) / \bar{\mu} \end{pmatrix}.
\]

Therefore, following a similar calculation as in Section 2.2, we can obtain the relationships between the components of \( \bar{F}_{1} \) and the components of \( \bar{F}^{1} \):

\[
\bar{F}_{1}^{+}(1,1) = \bar{F}^{1,+}(1,1), \quad \bar{F}_{1}^{+}(1,2) = \bar{F}^{1,+}(1,2) / \bar{\mu},
\]

\[
\bar{F}_{1}^{+}(2,1) = \bar{F}^{1,+}(2,1) \bar{\mu}, \quad \bar{F}_{1}^{+}(2,2) = \bar{F}^{1,+}(2,2).
\]

Based on these relationships, we rewrite the standard upwind scheme (3.25) as follows

\[
\bar{v}_{i+1,j} = \bar{v}_{i,j} - \frac{\Delta x}{\Delta y} \left( \bar{F}_{i,j}^{1,+}(1,1) (\bar{v}_{i,j} - \bar{v}_{i,j-1}) + \bar{F}_{i,j}^{1,+}(1,2) (\bar{p}_{i,j} - \bar{p}_{i,j-1}) / \bar{\mu}_{i,j} \\ + \bar{F}_{i,j}^{1,-}(1,1) (\bar{v}_{i,j+1} - \bar{v}_{i,j}) + \bar{F}_{i,j}^{1,-}(1,2) (\bar{p}_{i,j+1} - \bar{p}_{i,j}) / \bar{\mu}_{i,j} \right) \\
- \Delta x \bar{F}_{i,j}^{2}(1)
\]  

(3.26)

\[
\bar{p}_{i+1,j} = \bar{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( \bar{F}_{i,j}^{1,+}(2,1) (\bar{p}_{i,j} - \bar{p}_{i,j-1}) + \bar{F}_{i,j}^{1,+}(2,2) (\bar{p}_{i,j} - \bar{p}_{i,j-1}) \\ + \bar{F}_{i,j}^{1,-}(2,1) (\bar{p}_{i,j+1} - \bar{p}_{i,j}) + \bar{F}_{i,j}^{1,-}(2,2) (\bar{p}_{i,j+1} - \bar{p}_{i,j}) \right) \\
- \Delta x \bar{F}_{i,j}^{2}(2).
\]  

(3.27)

Remember that in Part (I) of this section, we derived the relationships between the
components of $\tilde{F}^1$ and the components of $F^1$:

$$
\tilde{F}^{1,\pm}(1, 1) = F^{1,\pm}(1, 1), \quad \tilde{F}^{1,\pm}(1, 2) = F^{1,\pm}(1, 2)(1 + \epsilon \tilde{\mu}), \\
\tilde{F}^{1,\pm}(2, 1) = F^{1,\pm}(2, 1)/(1 + \epsilon \tilde{\mu}), \quad \tilde{F}^{1,\pm}(2, 2) = F^{1,\pm}(2, 2).
$$

(3.28)

Substituting these relationships into the discretized equations (3.26) and (3.27), we arrive at

$$
\tilde{v}_{i+1,j} = \tilde{v}_{i,j} - \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(1, 1)(\tilde{v}_{i,j} - \tilde{v}_{i,j-1}) + F_{i,j}^{1,+}(1, 2)(1 + \epsilon \tilde{\mu}_{i,j})(\tilde{p}_{i,j} - \tilde{p}_{i,j-1})/\tilde{\mu}_{i,j} \\
+ F_{i,j}^{1,-}(1, 1)(\tilde{v}_{i,j+1} - \tilde{v}_{i,j}) + F_{i,j}^{1,-}(1, 2)(1 + \epsilon \tilde{\mu}_{i,j})(\tilde{p}_{i,j+1} - \tilde{p}_{i,j})/\tilde{\mu}_{i,j} \right) \\
- \Delta x \tilde{F}_{i,j}^2(1)
$$

(3.29)

$$
\tilde{p}_{i+1,j} = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(2, 1)(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1})/(1 + \epsilon \tilde{\mu}_{i+1,j}) + F_{i,j}^{1,+}(2, 2)(\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) \\
+ F_{i,j}^{1,-}(2, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j})/(1 + \epsilon \tilde{\mu}_{i+1,j}) + F_{i,j}^{1,-}(2, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \right) \\
- \Delta x \tilde{F}_{i,j}^2(2).
$$

(3.30)

The second step is to take the exponential of the first equation. Then finally we obtain

$$
\tilde{\mu}_{i+1,j} = \tilde{\mu}_{i,j} - \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(1, 1)(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + F_{i,j}^{1,+}(1, 2)(1 + \epsilon \tilde{\mu}_{i,j})(\tilde{p}_{i,j} - \tilde{p}_{i,j-1})/\tilde{\mu}_{i,j} \\
+ F_{i,j}^{1,-}(1, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j})/(1 + \epsilon \tilde{\mu}_{i+1,j}) + F_{i,j}^{1,-}(1, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j})/\tilde{\mu}_{i,j} \right) \\
- \Delta x \left( \tilde{F}_{i,j}^2(1, 1) + \tilde{F}_{i,j}^3(1)/\tilde{\mu}_{i,j} \right)
$$

(3.31)

$$
\tilde{p}_{i+1,j} = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(2, 1)(\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + F_{i,j}^{1,+}(2, 2)(\tilde{p}_{i,j} - \tilde{p}_{i,j-1}) \\
+ F_{i,j}^{1,-}(2, 1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j})/(1 + \epsilon \tilde{\mu}_{i+1,j}) + F_{i,j}^{1,-}(2, 2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j}) \right) \\
- \Delta x \left( \tilde{F}_{i,j}^2(2, 1) + \tilde{F}_{i,j}^3(2)/\tilde{\mu}_{i,j} \right)
$$

(3.32)

where we use the relationships in (3.24) to replace $\tilde{F}_{i,j}^2(1)$ by $\tilde{F}_{i,j}^2(1, 1) + \tilde{F}_{i,j}^3(1)/\tilde{\mu}_{i,j}$ in the first equation and $\tilde{F}_{i,j}^2(2)$ by $\tilde{F}_{i,j}^2(2, 1)\tilde{\mu}_{i,j} + \tilde{F}_{i,j}^3(2)$ in the second equation. If we apply the approximation $\exp(b\Delta x) \approx 1 + b\Delta x$ to the exponential terms that contain
the discretized pressure in the first equation and keep the remaining exponential terms unchanged, we obtain

\[
\tilde{\mu}_{i+1,j} = \left( \tilde{\mu}_{i,j} \left( 1 - \frac{\Delta x}{\tilde{\mu}_{i,j}} (F_{i,j}^{1,+}(1,1) - F_{i,j}^{1,-}(1,1)) \right) \right. \\
\left. \times \frac{\Delta x}{\tilde{\mu}_{i,j-1}} \cdot \frac{\Delta x}{\tilde{\mu}_{i,j+1}} \cdot \frac{\Delta x}{\tilde{\mu}_{i,j}} \cdot \frac{\Delta x}{\tilde{\mu}_{i,j}} \right)
\]

\[
- \frac{\Delta x}{\Delta y} \left( F_{i,j}^{1,+}(1,2)(\tilde{p}_{i,j} + \tilde{p}_{i,j-1}) + F_{i,j}^{1,-}(1,2)(\tilde{p}_{i,j+1} + \tilde{p}_{i,j}) \right) \left( 1 + \epsilon \tilde{\mu}_{i,j} \right)
\]

\[
\cdot \exp \left( - \frac{\Delta x}{\Delta x} \left( F_{i,j}^{2+}(1,1) + F_{i,j}^{3+}(1) \right) / \tilde{\mu}_{i,j} \right). \tag{3.33}
\]

We solve the above two discretized equations (3.32) and (3.33) with an alternating method: first solve (3.33) for \( \tilde{\mu}_{i+1,j} \) with \( \tilde{\mu}_{i,j} \) and \( \tilde{p}_{i,j} \), and then solve (3.32) for \( \tilde{p}_{i+1,j} \) with the computed values of \( \tilde{\mu}_{i,j} \), \( \tilde{\mu}_{i+1,j} \) and \( \tilde{p}_{i,j} \). Note that when we solve (3.33), we use \( \tilde{\mu}_{i,j} \) in the term \( 1 + \epsilon \tilde{\mu} \); while when we solve (3.32), we use \( \tilde{\mu}_{i+1,j} \) in the term \( 1 + \epsilon \tilde{\mu} \).

**(IV) The 2D Log-Elastographic Central Difference Scheme when the Nonlinear Term, \( \epsilon \tilde{\mu} \tilde{\nabla} \tilde{p} \), is Included:**

Similarly, if we discretize equations (3.21) and (3.22) with the central difference scheme, we get

\[
\tilde{v}_{i+1,j} = \frac{1}{2}(\tilde{v}_{i,j+1} + \tilde{v}_{i,j-1}) - \frac{\Delta x}{2\Delta y} \left( F_{i,j}^{1+}(1,1)(\tilde{v}_{i,j+1} - \tilde{v}_{i,j-1}) \right.
\]

\[
\left. + F_{i,j}^{1+}(2,2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1}) / \tilde{\mu}_{i,j} \right) - \Delta x \left( F_{i,j}^{2+}(1,1) + F_{i,j}^{3+}(1) / \tilde{\mu}_{i,j} \right) \tag{3.34}
\]

\[
\tilde{p}_{i+1,j} = \frac{1}{2}(\tilde{p}_{i,j+1} + \tilde{p}_{i,j-1}) - \frac{\Delta x}{2\Delta y} \left( F_{i,j}^{1+}(2,1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) \right.
\]

\[
\left. + F_{i,j}^{1+}(2,2)(\tilde{p}_{i,j+1} - \tilde{p}_{i,j-1}) \right) - \Delta x \left( F_{i,j}^{2+}(2,1) \tilde{\mu}_{i,j} + F_{i,j}^{3+}(2) \right). \tag{3.35}
\]

Taking the exponential of the first equation, and then approximating \( \exp (b\Delta x) \) by \( 1 + b\Delta x \) in the exponential terms that contain the pressure and keeping the remaining
exponential terms unchanged, we get

\[
\tilde{\mu}_{i+1,j} = \left( \frac{\tilde{\mu}_{i,j+1} \cdot \tilde{\mu}_{i,j-1}}{\Delta x} \right) \cdot \exp \left( -\Delta x \left( \bar{F}_{2}^{2}(i,j,1) + \bar{F}_{3}^{3}(1) / \tilde{\mu}_{i,j} \right) \right)
\]

(3.36)

where we have used the relationships in (3.15) to replace \( \bar{F}_{1}^{1}(1,1) \) by \( F^{1}(1,1) \), \( F^{1}(1,2) \) by \( F^{1}(1,2)(1 + \epsilon \tilde{\mu}) \), \( F^{1}(2,1) \) by \( F^{1}(2,1)/(1 + \epsilon \tilde{\mu}) \) and \( F^{1}(2,2) \) by \( F^{1}(2,2) \).

We solve the above two discretized equations (3.35) and (3.36) as follows: first solve (3.36) for \( \tilde{\mu}_{i+1,j} \), then solve (3.35) for \( \tilde{p}_{i+1,j} \).

Having established the 2D Log-Elastographic algorithms for the 2D plane strain elastic system both with and without the nonlinear term, we will next in Chapter 4, establish stability and accuracy for the 2D Log-Elastographic algorithms for the 2D plane strain elastic system with and without the nonlinear term.
CHAPTER 4
Stability and Accuracy

In this chapter, we establish stability and accuracy results for the $2D$ Log-Elastographic
nonlinear upwind and central difference schemes that calculate the pair $(\tilde{\mu}, p)$ utilizing the $2D$ plane strain elastic system. We will establish our results for the case
where we include the nonlinear term $\epsilon \tilde{\mu} \nabla p$ and for the case where we do not. These
are the schemes introduced in Chapter 3.

Before we present the proofs for our $2D$ Log-Elastographic schemes we present
the Von Neumann analysis for the partially implicit upwind and central difference
schemes for first order linear partial differential equation systems. The following
lemmas will help to prove the theorems in this chapter.

Lemma 4.1 Let $E$ be a diagonalizable matrix with real eigenvalues $\lambda^1$, $\lambda^2$, and

\[ G^1(\xi) = I - \frac{\Delta x}{\Delta y} \left( E^+(1 - e^{-k\xi}) + E^-(e^{k\xi} - 1) \right), \]

\[ G^2(\xi) = \frac{1}{2} \left( (e^{-k\xi} + e^{k\xi})I - \frac{\Delta x}{\Delta y} E(e^{k\xi} - e^{-k\xi}) \right), \]

where $k = \sqrt{-1}$, $E^\pm = S^{-1} \Lambda^\pm S$, $S$ is the eigenvector matrix of $E$, and $\Lambda^+$ ($\Lambda^-$) is
the eigenvalue matrix of $E$ with non-negative (negative) eigenvalues on its diagonal.
If $0 < \frac{\Delta x}{\Delta y} \max(|\lambda^1|, |\lambda^2|) \leq 1$, then the eigenvalues of $G^1$ and $G^2$ are all bounded
by 1, and hence $\|(G^1)^l\|_2 \leq C_1 C_2$, $\|(G^2)^l\|_2 \leq C_1 C_2$, for any $l$, where $C_1$, $C_2$ are
constants satisfying $\|S\|_2 \leq C_1$, $\|S^{-1}\|_2 \leq C_2$.

Proof. By hypothesis there exists a matrix $S$, which is the eigenvector matrix, such
that $\Lambda = SES^{-1}$ is a diagonal matrix with the real eigenvalues of $E$ on the diagonal.
We multiply $G^1$ on the left by $S$ and on the right by $S^{-1}$ to get

\[ H^1 = SG^1 S^{-1} = I - \frac{\Delta x}{\Delta y} \left( \Lambda^+(1 - e^{-k\xi}) + \Lambda^-(e^{k\xi} - 1) \right). \]
Since \( \frac{\Delta x}{\Delta y} \) max\(|\lambda^1|, |\lambda^2|\) \( \leq 1 \), if \( \lambda^{i^+} (\lambda^{i^-}) \) is a positive (negative) eigenvalue, then

\[
|\rho_j^+ (\xi)| = \left| 1 - \frac{\Delta x}{\Delta y} \lambda^{i^+} + \frac{\Delta x}{\Delta y} \lambda^{i^+} e^{-k\xi} \right| \leq 1
\]
or

\[
|\rho_j^- (\xi)| = \left| 1 + \frac{\Delta x}{\Delta y} \lambda^{i^-} - \frac{\Delta x}{\Delta y} \lambda^{i^-} e^{k\xi} \right| \leq 1,
\]

where \( \rho_j^+ , \rho_j^- \) are eigenvalues of \( H^1 \) corresponding to positive \( \lambda^{i^+} \) and negative \( \lambda^{i^-} \) of \( \Lambda \). Since the matrix \( (H^1)^*H^1 \) is a diagonal matrix with \( |\rho_j (\xi)|^2 \) on the diagonal and \( |\rho_j (\xi)|^2 \leq 1 \), we can derive that \( \| (H^1)^l \|_2 \leq 1 \), and hence for every \( l, l = 1, 2, ... \),

\[
\|(G^1)^l\|_2 = \|S^{-1}(H^1)^lS\|_2 \leq \|S^{-1}\|_2 \|S\|_2 \|(H^1)^l\|_2 \leq C_1 C_2,
\]

where \( C_1, C_2 \) are constants satisfying \( \|S\|_2 \leq C_1, \|S^{-1}\|_2 \leq C_2 \).

Similarly, if the eigenvalues of matrix \( E \) are \( \lambda \), then the eigenvalues of matrix \( G^2 \) are

\[
\eta = \frac{1}{2} (e^{k\xi} + e^{-k\xi}) - \frac{\Delta x}{2\Delta y} \lambda (e^{k\xi} - e^{-k\xi}) = \cos \xi - \left( \frac{\Delta x}{\Delta y} \lambda \sin \xi \right) k.
\]

Therefore, if \( \lambda \) is real and \( \Delta x/\Delta y|\lambda| \leq 1 \), then \( |\eta| \leq 1 \), and hence \( \|(G^2)^l\|_2 \leq C_1 C_2 \).

**Lemma 4.2** If \( \|G^l\|_2 \leq C_1 C_2, l = 1, 2, ... \), with \( C_1 C_2 \geq 1 \), then

\[
\|(G + \Delta xFG)^l\|_2 \leq C_1 C_2 e^{\|\Delta xC_1 F\|_2}.
\]

**Proof.** Consider the binomial expansion of \( (G + \Delta xFG)^l \), which has a \( G^l \) term, \( l \) terms containing one factor \( \Delta xFG \) and \( l - 1 \) factors which are equal to \( G \), \( \binom{l}{2} \) terms containing two factors which are \( \Delta xFG \)'s and \( l - 2 \) \( G \) factors, .... We look at this expansion in the following different cases:

\( j = 0 \): only one term \( G^l \) in this case, and \( \|G^l\|_2 \leq C_1 C_2 \).

\( j = 1 \): \( l \) terms, each one contains one factor \( \Delta xFG \) and \( l - 1 \) factors which are equal
The norm of each of these terms is bounded by \((C_1 C_2)^2 \Delta x \| F \|_2\), e.g.,
\[
\| G' \Delta x G \cdots G \|_2 \leq \| G \cdots G \|_2 \| \Delta x F \|_2 \| G \cdots G \|_2 \leq (C_1 C_2)^2 \Delta x \| F \|_2.
\]

For \(j = m - 1\): \(\begin{pmatrix} l \\ m - 1 \end{pmatrix}\) terms, each one contains \(m - 1\) factors of the form \(\Delta x G\)'s and \(l - (m - 1)\) factors which are equal to \(G\). It follows that each of these terms has the upper bound \((C_1 C_2)^m \Delta x^{m-1} \| F \|^{m-1}_2\).

For \(j = l\): \(\begin{pmatrix} l \\ l \end{pmatrix}\) term, which is \((\Delta x G)^l\) and satisfies
\[
\| (\Delta x G)^l \|_2 \leq \Delta x^l \| F \|_2 \| G \|_2 \leq (C_1 C_2)^l \Delta x^l \| F \|_2 \leq (C_1 C_2)^{l+1} \Delta x^l \| F \|_2.
\]

Adding all the bounds above together, we get
\[
\| (G + \Delta x G)^l \|_2 \leq \sum_{j=0}^l \binom{l}{j} (C_1 C_2)^{j+1} \Delta x^j \| F \|_2^j
\]
\[
\leq C_1 C_2 \sum_{j=0}^l \frac{(lC_1 C_2 \Delta x \| F \|_2)^j}{j!}
\]
\[
\leq C_1 C_2 e^{\Delta x C_1 C_2 \| F \|_2}.
\]

\[\square\]

### 4.1 Stability Analysis for the Partially Implicit Upwind and Central Difference Schemes in the \(L_2\) norm

First we apply the Von Neumann analysis to establish the stability in the \(L_2\) norm for the partially implicit upwind and central difference schemes given in (3.6) and (3.7). We adopt the freezing coefficients assumption and so omit the discretization notation for the coefficients. So strictly speaking this analysis applies when the coefficients in our differential equations are constant. Our relabeling in this section is then \(E_{i,j}^1 = E^1, E_{i,j}^2 = E^2\), etc.
**Theorem 4.1:** Consider the partially implicit upwind scheme and assume: \(E^1\) is diagonalizable; the eigenvalues \(\lambda^1, \lambda^2\) of \(E^1\) are real; the positive step sizes \(\Delta x\) and \(\Delta y\) satisfy \(0 < (\Delta x/\Delta y) \max(|\lambda^1|,|\lambda^2|) \leq 1\); and \((1 + \Delta x E^2(1,1)) > 0\). Then the discretized solution \(V_{i,j}\) of

\[
V_{i+1,j} = (I + \Delta x E^2)^{-1} \left( \left( I - \frac{\Delta x}{\Delta y} E^{1,+} + \frac{\Delta x}{\Delta y} E^{1,-} \right) V_{i,j} 
+ \frac{\Delta x}{\Delta y} E^{1,+} V_{i,j-1} - \frac{\Delta x}{\Delta y} E^{1,-} V_{i,j+1} - \Delta x E^3 \right),
\]

(4.1)
satisfies

\[
\|V_{n+1}\|_2 \leq \max(C_1 C_2, 1) \left( \frac{e^{|X R|}}{R^{n+1}} \|V_0\|_2 
+ \Delta x e^{X |R| (n+1) - 1} \| (I + \Delta x E^2)^{-1} \|_2 \|E^3\|_2 \right),
\]

where \(R, C_1, C_2\) and \(\alpha\) are positive constants and \(X = (n + 1) \Delta x\).

**Remark:** The constants \(R, C_1, C_2\) and \(\alpha\) can be estimated in terms of the coefficient matrices. We can establish that

\[
R = |1 + E^2(1,1)\Delta x|, \quad \alpha = C_1 C_2 (|E^2(1,1)|^2 + |E^2(2,1)|^2)^{\frac{1}{2}}
\]

where the eigenvector matrix \(S \) of \(E^1\), satisfies \(\|S\|_2 \leq C_1, \|S^{-1}\|_2 \leq C_2\), with

\[
C_1 = \max(t^1, t^2), \quad \text{where } t^1, t^2 \text{ are the singular values of } S,
\]

\[
C_2 = \max(t^1, t^2)/\bar{C}_2 \quad \text{and}
\]

\[
\bar{C}_2 = \min_{i,j} |E_{i,j}^1(1,2) E_{i,j}^1(2,1) - \lambda_{i,j}^1 \lambda_{i,j}^2 + E_{i,j}^1(1,1) \lambda_{i,j}^2 + E_{i,j}^1(2,2) \lambda_{i,j}^1 - E_{i,j}^1(1,1) E_{i,j}^1(2,2)|.
\]

**Proof.** To apply the Von Neumann stability analysis we let

\[
\dot{V}_{i+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-k^2 \xi^2} V_{i+1,j}
\]

(4.2)
where $k = \sqrt{-1}$. Assuming the coefficient matrices are constant we then write

$$
\hat{V}_{i+1}(\xi) = (I + \Delta x E^2)^{-1} (G(\xi)\hat{V}_i(\xi) - \Delta x E^3 e^{-kj\xi})
$$

(4.3)

where $G(\xi)$ is a $2 \times 2$ matrix given by

$$
G(\xi) = \begin{pmatrix}
I - \frac{\Delta x}{\Delta y} E^{1,+} + \frac{\Delta x}{\Delta y} E^{1,-} \\
+\frac{\Delta x}{\Delta y} E^{1,+}e^{-k\xi} - \frac{\Delta x}{\Delta y} E^{1,-}e^{k\xi}
\end{pmatrix}
$$

So after $n+1$ iterations,

$$
\hat{V}_{n+1} = (I + \Delta x E^2)^{-1} G(\xi) \hat{V}_n - \Delta x (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi}
$$

$$
= (I + \Delta x E^2)^{-1} G(\xi) ((I + \Delta x E^2)^{-1} G(\xi) \hat{V}_{n-1} - \Delta x (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi})
$$

$$
- \Delta x (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi}
$$

$$
= ((I + \Delta x E^2)^{-1} G(\xi))^2 \hat{V}_{n-1}
$$

$$
- \Delta x ((I + \Delta x E^2)^{-1} G(\xi) + I) (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi}
$$

$$
= \ldots
$$

$$
= ((I + \Delta x E^2)^{-1} G(\xi))^{n+1} \hat{V}_0
$$

$$
- \Delta x \left(((I + \Delta x E^2)^{-1} G(\xi))^n + \cdots + I\right) (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi},
$$

implying

$$
\|\hat{V}_{n+1}\|_2 \leq \|((I + \Delta x E^2)^{-1} G)^{n+1}\|_2 \|\hat{V}_0\|_2 + \Delta x \left(\|((I + \Delta x E^2)^{-1} G)^n\|_2 + \|((I + \Delta x E^2)^{-1} G)^{n-1}\|_2 + \cdots + 1\right)\|(I + \Delta x E^2)^{-1}\|_2 \|E^3\|_2.
$$

In order to establish our stability result, we need to find non-negative constants $K$ and $\alpha$ such that

$$
\|((I + \Delta x E^2)^{-1} G)^l\|_2 \leq K e^{\alpha l \Delta x}
$$

(4.4)

for any $l$. 

From Lemma 4.1, $\|G\|_2 \leq C_1C_2$, where $C_1, C_2$ are constants satisfying $\|S\|_2 \leq C_1, \|S^{-1}\|_2 \leq C_2$. Furthermore, in our case, $E^2$ is in the form of $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$, where $a = E^2(1, 1)$, $c = E^2(2, 1)$. Therefore,

$$(I + \Delta xE^2)^{-1} = \frac{1}{1 + a\Delta x} \begin{pmatrix} 1 & 0 \\ -c\Delta x & 1 + a\Delta x \end{pmatrix} = \frac{1}{1 + a\Delta x} (I + \Delta xF)$$

where $F = \begin{pmatrix} 0 & 0 \\ -c & a \end{pmatrix}$. And hence

$$(I + \Delta xE^2)^{-1}G^l = \left(\frac{1}{1 + a\Delta x}\right)^l (G + \Delta xFG)^l.$$  

From Lemma 4.2, we know that

$$\|(G + \Delta xFG)^l\|_2 \leq C_1C_2e^{\Delta xC_1C_2\|F\|_2},$$

Therefore,

$$\|(I + \Delta xE^2)^{-1}G^l\|_2 \leq \frac{1}{|1 + a\Delta x|^l} C_1C_2e^{\Delta xC_1C_2\|F\|_2},$$

for any $l$, where $\|F\|_2 = (|E^2(1, 1)|^2 + |E^2(2, 1)|^2)^\frac{1}{2}$. And hence

$$\|\hat{V}_{n+1}\|_2 \leq \frac{C_1C_2}{1 + a\Delta x}e^{C_1C_2\|F\|_2\Delta x} \|\hat{V}_0\|_2$$

$$+ \Delta x \frac{C_1C_2}{|1 + a\Delta x|^l} e^{C_1C_2\|F\|_2\Delta x} \frac{1}{|1 + a\Delta x|} (\|I + \Delta xE^2\|_2)^{-1} \|E^3\|_2.$$  

$\blacksquare$

**Remark:** The idea in this proof and in the proof of the next theorem follows the proof in [56].

We will utilize a similar proof for the partially implicit central difference
scheme. To do this, we define
\[ \tilde{G}(\xi) = \frac{1}{2} \left( I - \frac{\Delta x}{\Delta y} E^1 \right) e^{k\xi} + \frac{1}{2} \left( I + \frac{\Delta x}{\Delta y} E^1 \right) e^{-k\xi}. \]

From Lemma 4.1, \( \|\tilde{G}\|_2 \leq C_1 C_2 \), where \( C_1, C_2 \) are constants satisfying \( \|S\|_2 \leq C_1 \), \( \|S^{-1}\|_2 \leq C_2 \). Therefore, applying again the method of Fourier stability analysis but now to the partially implicit central difference scheme we arrive at
\[ \hat{V}_{i+1}(\xi) = (I + \Delta x E^2)^{-1} (\tilde{G}(\xi) \hat{V}_i(\xi) - \Delta x E^3 e^{-k\xi}). \] (4.5)

Following the same steps as in the proof of Theorem 4.1 we can find non-negative constants \( K \) and \( \alpha \) such that
\[ \|((I + \Delta x E^2)^{-1} \tilde{G})^l\|_2 \leq Ke^{\alpha l \Delta x} \] (4.6)
for any \( l \), which leads to the following theorem.

**Theorem 4.2:** Consider the partially implicit central difference scheme and assume: \( E^1 \) is diagonalizable; the eigenvalues \( \lambda^1, \lambda^2 \) of \( E^1 \) are real; the step sizes \( \Delta x \) and \( \Delta y \) satisfy \( (\Delta x/\Delta y) \max(|\lambda^1|, |\lambda^2|) \leq 1 \); and \( 1 + \Delta x E^2(1,1) > 0 \). Then the discretized solution \( V_{i,j} \) of
\[ V_{i+1,j} = (I + \Delta x E^2)^{-1} \left( \frac{1}{2} \left( I - \frac{\Delta x}{\Delta y} E^1 \right) V_{i,j+1} + \frac{1}{2} \left( I + \frac{\Delta x}{\Delta y} E^1 \right) V_{i,j-1} - \Delta x E^3 \right) \] (4.7)
satisfies
\[ \|V_{n+1}\|_2 \leq \max(C_1 C_2, 1) \left( \frac{e^{\alpha X}}{R^{n+1}} \|V_0\|_2 + \Delta x e^{\alpha X} R^{-(n+1)} - 1 - \frac{1}{(e^{\alpha X} / R) - 1} \|((I + \Delta x E^2)^{-1})_2 \|_2 \|E^3\|_2 \right), \]
where \( R, C_1, C_2 \) and \( \alpha \) are the same constants as in Theorem 4.1 and \( X = (n+1)\Delta x \).
4.2 Stability Analysis for the 2D Log-Elastographic Methods for the Elastic System without the $\epsilon\tilde{\mu}\nabla p$ Term

In this section, we establish the stability results for the 2D Log-Elastographic upwind and central difference methods for the elastic system without the $\epsilon\tilde{\mu}\nabla p$ term. Recall that the 2D Log-Elastographic upwind scheme contains one nonlinear discretized equation and one linear discretized equation; the nonlinear discretized equation is obtained by taking the exponential of the first discretized equation in the following discretized system for $w = (\ln\tilde{\mu}, \hat{\tilde{p}})^T$

$$W_{i+1,j} = W_{i,j} - \frac{\Delta x}{\Delta y} \left( A_{i,j}^+ (W_{i,j} - W_{i,j-1}) + A_{i,j}^- (W_{i,j+1} - W_{i,j}) \right) - \Delta x E_{i,j}^\mu$$  \hspace{1cm} (4.8)

where $W_{i,j}$ is the approximated value of $w$ at $(i\Delta x, j\Delta y)$,

$$A_{i,j}^\mu = \begin{pmatrix} E_{i,j}^1(1,1) & E_{i,j}^1(1,2)/\tilde{\mu}_{i,j} \\ E_{i,j}^1(2,1)\tilde{\mu}_{i,j} & E_{i,j}^1(2,2) \end{pmatrix}$$

$$E_{i,j}^\mu = \begin{pmatrix} E_{i,j}^2(1,1) + E_{i,j}^3(1)/\tilde{\mu}_{i,j} \\ E_{i,j}^2(2,1)\tilde{\mu}_{i,j} + E_{i,j}^3(2) \end{pmatrix}.$$

Thus, to establish the stability for the 2D Log-Elastographic upwind algorithm, we first apply the Von Neumann Analysis to establish the stability of the discretized system (4.8) for $(\ln\tilde{\mu}, \hat{\tilde{p}})^T$, and then we show the stability of the 2D Log-Elastographic upwind scheme for $(\tilde{\mu}, \hat{\tilde{p}})^T$. Here, we also adopt the freezing coefficients assumption. But since the coefficient matrices in (4.8) contain the unknown $\tilde{\mu}$, when we adopt the freezing coefficients assumption, we freeze the components of the matrices $E^i$, $i = 1, 2, 3$ in the coefficient matrices $A^\mu$ and $E^\mu$ in both $x$ and $y$ directions but the unknown $\tilde{\mu}$ in the coefficient matrices $A^\mu$ and $E^\mu$ only in $y$ direction. We will call this value $\tilde{\mu}^x$. This choice can be thought of as the average value of $\tilde{\mu}$ in $y$ when $x$ is a fixed value. This is the approach we take in the proof of Theorem 4.3. This is equivalent to setting $\xi = 0$ in the Fourier transform of $\tilde{\mu}$. On the other hand, we can also apply a similar idea as in [23]. That is, we assume that the stability of
a nonlinear finite difference scheme is governed by the local amplification matrix. This is equivalent to considering the stability of (4.8) when its coefficient matrices are constant locally. Under this assumption, we can thereby freeze the whole coefficient matrices including the unknown part. We will give comparison of these two methods after Theorem 4.3. By freezing the coefficient matrices, \( E_1, E_2 \), we can then omit the discretization notation for them. Our relabeling in this section is thus again \( E_{1,i,j} = E_1, E_{2,i,j} = E_2 \), etc.

**Theorem 4.3:** Consider the following 2D Log-Elastographic upwind scheme

\[
\tilde{\mu}_{i+1,j} = \left( \tilde{\mu}_{i,j} + \frac{\Delta x}{\Delta y} \left( E^{1,+}(1,1) \left( \tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1} \right) + E^{1,-}(1,2) \left( \tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j} \right) \right) \right) \cdot \mu_{i,j-1} \cdot \mu_{i,j+1} \\
- \Delta x \left( E^{2,+}(1,2) \left( \tilde{p}_{i+1,j} - \tilde{p}_{i,j-1} \right) + E^{2,-}(2,1) \left( \tilde{p}_{i,j+1} - \tilde{p}_{i,j} \right) \right) \cdot \exp \left( - \Delta x (E^2(1,1) + E^3(1)/\tilde{\mu}_{i,j}) \right) \\
\tilde{p}_{i+1,j} = \tilde{p}_{i,j} - \frac{\Delta x}{\Delta y} \left( E^{1,+}(2,1) \left( \tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1} \right) + E^{1,-}(2,2) \left( \tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j} \right) \right) \\
+ E^{1,+}(2,2) \left( \tilde{p}_{i,j} - \tilde{p}_{i,j-1} \right) + E^{1,-}(2,2) \left( \tilde{p}_{i,j+1} - \tilde{p}_{i,j} \right) \right) \\
- \Delta x \left( E^2(2,1) \tilde{\mu}_{i,j} + E^3(2) \right)
\]

(4.9)

(4.10)

and assume: (1) the eigenvalues \( \lambda^1, \lambda^2 \) of the matrix \( E_1 \) are real; (2) the step sizes \( \Delta x \) and \( \Delta y \) satisfy \( (\Delta x/\Delta y) \max(|\lambda^1|, |\lambda^2|) \leq 1 \); (3) the scaled shear modulus \( \tilde{\mu}_{i,j} \) is positive and has a lower bound \( L_\mu \), an upper bound \( U_\mu \) and the numerical solution \( \tilde{\mu}_{i,j} \) has a lower bound \( L_\mu/2 \), an upper bound \( 2U_\mu \) for all \( i \) and \( j \); (4) the initial values \( \tilde{\mu}_0 \) are real and satisfy \( |\tilde{\mu}_0| \geq 1 \); and (5) the total variation \( TV(\tilde{\mu}^x) \) is bounded. Then the discretized solution \( \tilde{\mu}_{i,j} \) and \( \tilde{p}_{i,j} \) satisfy

\[
\left( \prod_{j=1}^{N} \tilde{\mu}_{n+1,j} \right)^{1/N} \leq e^{\sqrt{2c/N}} (\prod_{j=1}^{N} \mu_{0,j})^{\sqrt{2c/N}}, \| \tilde{p}_{n+1} \|_2 \leq c \| \ln \tilde{\mu}_0 \|_2 + \bar{c},
\]

(4.11)

where \( c \) and \( \bar{c} \) are constants.

**Remark 1:** The constants \( c \) and \( \bar{c} \) can be estimated in terms of the coefficient matrices. If we define \( C_1, C_2 \) to satisfy \( \| S \|_2 \leq C_1, \| S^{-1} \|_2 \leq C_2 \), where \( S \) is the
eigenvector matrix of $E^1$, then
\[ c = \max(\bar{C}_1\bar{C}_2, 1)e^{\gamma}, \quad \bar{c} = \max(\bar{C}_1\bar{C}_2, 1)(n+1)\Delta xe^{\gamma}\bar{C}, \]
where $\bar{C}_i = \max(2/L\tilde{\mu}, 2U\tilde{\mu})C_i$, $i = 1, 2$, $\gamma = \frac{TV(\tilde{\mu})}{|d|}\frac{|U\tilde{\mu}|}{L\tilde{\mu}}\max(|S(1,1)|, |S(2,2)|)C_1$,
\[ d \text{ is the determinant of the matrix } S, \]
and
\[ \bar{C} = \left((|E^2(1,1)| + |E^3(1)|/(2L\tilde{\mu}))^2 + (2|E^2(2,1)|U\tilde{\mu} + |E^3(2)|)^2\right)^{1/2}. \]

**Proof.** We first consider the stability for the discretized system (4.8). This scheme is similar to the standard upwind scheme except that its coefficient matrices contain the unknown. Apply the Von Neumann stability analysis by freezing the coefficients in $E^1$, $E^2$ in $x$ and $y$ and the unknown $\tilde{\mu}$ in $y$, and let
\[ \hat{W}_{i+1}(\xi) = \frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-kj\xi}W_{i+1,j} \]
where $k = \sqrt{-1}$. Then we can obtain
\[ \hat{W}_{i+1}(\xi) = G\hat{\mu}(\xi)\hat{W}_i(\xi) - \Delta x E\hat{\mu}(\xi)e^{-kj\xi} \]
where we have approximated the unknown $\hat{\mu}_{i,j}$ in the coefficient matrices by $\hat{\mu}^x$, for all $j$, e.g. the average value of $\hat{\mu}_{i,j}$ over $j$, which is equivalent to setting $\xi = 0$ in the Fourier transform of $\hat{\mu}_{i,j}$. And hence,
\[ G\hat{\mu}(\xi) = \left(I - \frac{\Delta x}{\Delta y}A_{\hat{\mu}^+,+} + \frac{\Delta x}{\Delta y}A_{\hat{\mu}^+,+} + \frac{\Delta x}{\Delta y}A_{\hat{\mu}^+,+}e^{-k\xi} - \frac{\Delta x}{\Delta y}A_{\hat{\mu}^+,+}e^{k\xi} \right) \]
where $A_{\hat{\mu}^x,\pm}$ is the positive (negative) part of the coefficient matrix $A_{\hat{\mu}^x}$ which con-
tains the positive (negative) eigenvalues of $A^{\tilde{\mu}_i}$. After $n+1$ iterations,

$$\hat{W}_{n+1} = G^{\tilde{\mu}_n}(\xi) \hat{W}_n - \Delta x E^{\tilde{\mu}_n} e^{-k_j \xi}$$

$$= G^{\tilde{\mu}_n}(\xi) \left( G^{\tilde{\mu}_{n-1}}(\xi) \hat{W}_{n-1} - \Delta x E^{\tilde{\mu}_{n-1}} e^{-k_j \xi} \right) - \Delta x E^{\tilde{\mu}_n} e^{-k_j \xi}$$

$$= G^{\tilde{\mu}_n}(\xi) G^{\tilde{\mu}_{n-1}}(\xi) \left( G^{\tilde{\mu}_{n-2}}(\xi) \hat{W}_{n-2} - \Delta x E^{\tilde{\mu}_{n-2}} e^{-k_j \xi} \right)$$

$$- \Delta x \left( G^{\tilde{\mu}_n}(\xi) E^{\tilde{\mu}_{n-1}} + E^{\tilde{\mu}_n} \right) e^{-k_j \xi}$$

$$= \ldots$$

$$= \prod_{m=0}^{n} G^{\tilde{\mu}_m}(\xi) \hat{W}_0 - \Delta x \left( \prod_{m=1}^{n} G^{\tilde{\mu}_m}(\xi) E^{\tilde{\mu}_m} + \prod_{m=2}^{n} G^{\tilde{\mu}_m}(\xi) E^{\tilde{\mu}_m} + \ldots \right.$$  

$$+ \left. G^{\tilde{\mu}_n} E^{\tilde{\mu}_{n-1}} + E^{\tilde{\mu}_n} \right) e^{-k_j \xi},$$

$$\| \hat{W}_{n+1} \|_2 \leq \| \prod_{m=0}^{n} G^{\tilde{\mu}_m}(\xi) \hat{W}_0 \|_2 + \Delta x \left( \prod_{m=1}^{n} G^{\tilde{\mu}_m}(\xi) \| E^{\tilde{\mu}_m} \|_2 \right.$$  

$$+ \left. \prod_{m=2}^{n} G^{\tilde{\mu}_m}(\xi) \| E^{\tilde{\mu}_m} \|_2 + \ldots + \| G^{\tilde{\mu}_n} \|_2 \| E^{\tilde{\mu}_{n-1}} \|_2 + \| E^{\tilde{\mu}_n} \|_2 \right).$$

implying that

$$\| \hat{W}_{n+1} \|_2 \leq \| \prod_{m=0}^{n} G^{\tilde{\mu}_m}(\xi) \|_2 \| \hat{W}_0 \|_2 + \Delta x \left( \prod_{m=1}^{n} G^{\tilde{\mu}_m}(\xi) \| E^{\tilde{\mu}_m} \|_2 \right.$$  

$$+ \left. \prod_{m=2}^{n} G^{\tilde{\mu}_m}(\xi) \| E^{\tilde{\mu}_m} \|_2 + \ldots + \| G^{\tilde{\mu}_n} \|_2 + 1 \right) \max_m \| E^{\tilde{\mu}_m} \|_2. \quad (4.13)$$

In order to establish the stability results, we need to find non-negative constants $\bar{K}$ and $\bar{\alpha}$ such that

$$\| \prod_{m=l}^{n} G^{\tilde{\mu}_m}(\xi) \|_2 \leq \bar{K} e^{\bar{\alpha}(n-l+1) \Delta x} \quad (4.14)$$

for any $l$.

From Chapter 2 we know that $A^{\tilde{\mu}_i}$ and $E^1$ have the same eigenvalue matrix

$$\Lambda = \begin{pmatrix} \lambda^1 & 0 \\ 0 & \lambda^2 \end{pmatrix},$$

and the eigenvector matrix $S^{\tilde{\mu}_i}$ of $A^{\tilde{\mu}_i}$ and its inverse matrix
\((S\tilde{\mu}_i)^{-1}\) are

\[
S\tilde{\mu}_i = \begin{pmatrix}
S(1,1)/\tilde{\mu}_i & S(1,2) \\
S(2,1) & S(2,2)/\tilde{\mu}_i
\end{pmatrix}, \quad (S\tilde{\mu}_i)^{-1} = \frac{1}{d} \begin{pmatrix}
S(2,2)/\tilde{\mu}_i & -S(1,2) \\
-S(2,1) & S(1,1)/\tilde{\mu}_i
\end{pmatrix}
\]

where \(S(i,j)\) is the component of the eigenvector matrix \(S\) of \(E^1\) at the \(i^{th}\) row and the \(j^{th}\) column, and \(d\) is the determinant of \(S\). Multiplying \(G\tilde{\mu}_i\) on the left by \(S\tilde{\mu}_i\) and on the right by \((S\tilde{\mu}_i)^{-1}\), we get

\[
H = S\tilde{\mu}_i G\tilde{\mu}_i (S\tilde{\mu}_i)^{-1} = I - \frac{\Delta x}{\Delta y} \left( \Lambda^+(1 - e^{-k\xi}) + \Lambda^-(e^{k\xi} - 1) \right).
\]

From Lemma 4.1, we know that if \(\Delta x/\Delta y \max(|\lambda_1|, |\lambda_2|) \leq 1\), then \(\|H\|_2 \leq 1\), and hence \(\|H\|_2^{n-l+1} \leq 1\) for any \(l\). Therefore,

\[
\left\| \prod_{m=l}^{n} G\tilde{\mu}_m(\xi) \right\|_2 = \left\| \prod_{m=l}^{n} (S\tilde{\mu}_m)^{-1} H S\tilde{\mu}_m \right\|_2
\leq \| (S\tilde{\mu}_l)^{-1} \|_2 \| S\tilde{\mu}_l \|_2 \| H \|_2^{n-l+1} \prod_{m=l}^{n-1} \| S\tilde{\mu}_m (S\tilde{\mu}_{m+1})^{-1} \|_2
\leq \bar{C}_1 \bar{C}_2 \prod_{m=l}^{n-1} \| S\tilde{\mu}_m (S\tilde{\mu}_{m+1})^{-1} \|_2
\]

where \(\bar{C}_i = \max(2/L\tilde{\mu}, 2U\tilde{\mu})C_i, \ i = 1, 2\), and \(C_1, C_2\) are constants satisfying \(\|S\|_2 \leq C_1, \|S^{-1}\|_2 \leq C_2\).
Next we derive an upper bound for \( \| S\tilde{\mu}_m (S\tilde{\mu}_{m+1})^{-1} \|_2 \) for any \( m = l, \ldots, n-1 \).

\[
S\tilde{\mu}_m (S\tilde{\mu}_{m+1})^{-1} = \frac{1}{d} \left( \begin{array}{ccc} S(1,1)/\tilde{\mu}_m^x & S(1,2) \\ S(2,1) & S(2,2)/\tilde{\mu}_m^x \end{array} \right) . \left( \begin{array}{ccc} S(2,2)\tilde{\mu}_{m+1} - S(1,2) \\ -S(2,1) & S(1,1)/\tilde{\mu}_m^x \end{array} \right)
\]

\[
= \frac{1}{d} \left( \begin{array}{ccc} S(1,1)S(2,2)\frac{\tilde{\mu}_{m+1}}{\tilde{\mu}_m} - S(1,2)S(2,1) & S(1,1)S(1,2)(\frac{1}{\tilde{\mu}_{m+1}} - \frac{1}{\tilde{\mu}_m}) \\ S(2,1)S(2,2)(\tilde{\mu}_{m+1} - \tilde{\mu}_m) & S(1,1)S(2,2)\frac{\tilde{\mu}_m}{\tilde{\mu}_{m+1}} - S(1,2)S(2,1) \end{array} \right)
\]

\[
= I + \frac{1}{d} \left( \begin{array}{ccc} S(1,1)S(2,2)(\frac{\tilde{\mu}_{m+1}}{\tilde{\mu}_m} - 1) & S(1,1)S(1,2)(\frac{1}{\tilde{\mu}_{m+1}} - \frac{1}{\tilde{\mu}_m}) \\ S(2,1)S(2,2)(\tilde{\mu}_{m+1} - \tilde{\mu}_m) & S(1,1)S(2,2)(\frac{\tilde{\mu}_m}{\tilde{\mu}_{m+1}} - 1) \end{array} \right)
\]

\[
= I + \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d\tilde{\mu}_{m+1}\tilde{\mu}_m} D_m
\]

where \( D_m = \left( \begin{array}{ccc} S(1,1)S(2,2)\tilde{\mu}_{m+1}^x & -S(1,1)S(1,2) \\ S(2,1)S(2,2)\tilde{\mu}_m^x\tilde{\mu}_{m+1}^x & -S(1,1)S(2,2)\tilde{\mu}_m^x \end{array} \right) \), and

\[
\left\| \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d\tilde{\mu}_{m+1}\tilde{\mu}_m} D_m \right\|_2 \leq \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d} \frac{1}{4|L|} \| D_m \|_2
\]

\[
\leq \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d} \frac{1}{4|L|} \| D_m \|_F
\]

\[
\leq \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d} \left( \frac{|U|}{|L|} \right)^2 \max(|S(1,1)|, |S(2,2)|) \| S \|_F
\]

\[
\leq \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d} \left( \frac{|U|}{|L|} \right)^2 \max(|S(1,1)|, |S(2,2)|) \sqrt{2} C_1,
\]

where \( \| S \|_F \) is the Frobenius norm of \( S \). Thus,

\[
\| S\tilde{\mu}_m (S\tilde{\mu}_{m+1})^{-1} \|_2 = \left\| I + \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d\tilde{\mu}_{m+1}\tilde{\mu}_m} D_m \right\|_2 \leq 1 + \left\| \frac{\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d\tilde{\mu}_{m+1}\tilde{\mu}_m} D_m \right\|_2,
\]
and hence
\[
\prod_{m=l}^{n-1} \left\| S^{\tilde{\mu}_m} (S^{\tilde{\mu}_{n+1}})^{-1} \right\|_2 \leq \prod_{m=l}^{n-1} \left( 1 + \left\| \frac{d\tilde{\mu}_{m+1} - \tilde{\mu}_m}{d\tilde{\mu}_{m+1}} D_m \right\|_2 \right) \\
\leq \prod_{m=l}^{n-1} e^{\frac{\|\tilde{\mu}_{m+1} - \tilde{\mu}_m\|_2}{\|\tilde{\mu}_{m+1} - \tilde{\mu}_m\|_2}} D_m \|
\leq e^{\sum_{m=l}^{n-1} \frac{\|\tilde{\mu}_{m+1} - \tilde{\mu}_m\|_2}{\|\tilde{\mu}_{m+1} - \tilde{\mu}_m\|_2}} D_m ^2 \leq e^\gamma
\]

where \( \gamma = TV(\tilde{\mu}^x) \frac{U_{\tilde{\mu}}}{L_{\tilde{\mu}}} \max(|S(1, 1)|, |S(2, 2)|) C_1 / |d| \). Therefore,
\[
\| \tilde{W}_{n+1} \|_2 \leq \max(\tilde{C}_1 \tilde{C}_2, 1) e^\gamma \left( \| \tilde{W}_0 \|_2 + (n + 1) \Delta x \tilde{C} \right),
\]

where \( \tilde{C} = ((|E^2(1, 1)| + |E^3(1)|/(2L_{\tilde{\mu}}))^2 + (2|E^2(2, 1)|U_{\tilde{\mu}} + |E^3(2)|)^2)^{1/2} \). Since we set the initial value \( \tilde{p}_0 \) to be zero, this means that
\[
\| \ln \tilde{\mu}_{n+1} \|_2 \leq \| \tilde{W}_{n+1} \|_2 \leq \max(\tilde{C}_1 \tilde{C}_2, 1) e^\gamma \left( \| \ln \tilde{\mu}_0 \|_2 + (n + 1) \Delta x \tilde{C} \right),
\]

\[
\| \tilde{p}_{n+1} \|_2 \leq \| \tilde{W}_{n+1} \|_2 \leq \max(\tilde{C}_1 \tilde{C}_2, 1) e^\gamma \left( \| \ln \tilde{\mu}_0 \|_2 + (n + 1) \Delta x \tilde{C} \right),
\]

where note that \( \| \ln \tilde{\mu}_{n+1} \|_2 ^2 + \| \tilde{p}_{n+1} \|_2 ^2 = \| \tilde{W}_{n+1} \|_2 ^2 \).

Lastly, we show that \( \tilde{\mu}_{n+1,j} \) can also be bounded. From
\[
\ln |a| \leq |\ln a| \Rightarrow |a| \leq e^{\ln a}
\]

for any complex number \( a \), and
\[
\sum_{j=1}^{N} |\ln \tilde{\mu}_{n+1,j}| \leq \sqrt{2} \| \ln \tilde{\mu}_{n+1} \|_2 \leq \sqrt{2} (c \| \ln \tilde{\mu}_0 \|_2 + \bar{c}),
\]

where \( c = \max(\tilde{C}_1 \tilde{C}_2, 1) e^\gamma \) and \( \bar{c} = \max(\tilde{C}_1 \tilde{C}_2, 1)(n + 1) \Delta x e^\gamma \tilde{C} \), we can derive that
\[
\prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \leq \prod_{j=1}^{N} e^{\| \ln \tilde{\mu}_{n+1,j} \|} = e^{\sum_{j=1}^{N} |\ln \tilde{\mu}_{n+1,j}|} \leq e^{\sqrt{2} \bar{c}} \cdot e^{\sqrt{2} \| \ln \tilde{\mu}_0 \|_2}.
\]
So if \( \tilde{\mu}_{0,j} \) are real and satisfy \( \tilde{\mu}_{0,j} \geq 1 \) for all \( j \) and since \( (\sum_{j=1}^{n} a_j^2)^{1/2} \leq \sum_{j=1}^{n} a_j \) when \( a_j \geq 0 \) for any \( j \), then \( \prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \leq e^{\sqrt{2c}} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2c}} \). Taking the \( N \)th root of both sides we obtain that the geometric mean of \( \tilde{\mu}_{n+1} \) satisfies

\[
\left( \prod_{j=1}^{N} \tilde{\mu}_{n+1,j} \right)^{1/N} \leq e^{\sqrt{2c}/N} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2c}/N}.
\]

This finishes our proof.

**Remark 2:** Note that \( \tilde{c}^2 = (|E^2(1,1)| + |E^3(1)|/(2L_{\tilde{\mu}}))^2 + (2|E^2(2,1)|U_{\tilde{\mu}} + |E^3(2)|)^2 \) is an upper bound for the sum \( (E^2(1,1) + E^3(1)/\tilde{\mu})^2 + (E^2(2,1)\tilde{\mu} + E^3(2))^2 = (|E^\tilde{\mu}|)^2 \). In the regions where the scaled shear modulus \( \tilde{\mu} \) is nearly constant and the pressure is nearly constant, then the two quantities \( E^2(1,1) + E^3(1)/\tilde{\mu} \) and \( E^2(2,1)\tilde{\mu} + E^3(2) \) are nearly equal to zero. This shows that \( (|E^\tilde{\mu}|)^2 \) is actually a small value in those regions. So while we make a uniform bound for \( \|E^\tilde{\mu}\|_2 \), \( m = 0, 1, ..., n \) in (4.13), more careful analysis for a specific \( \tilde{\mu} \) which is constant throughout subregions, may produce a much smaller bound. This also explains in part the success of the 2D Log-Elastographic algorithm.

**Remark 3:** We can also apply a similar idea as in [23], that is a fixed constant value of \( \tilde{\mu} \) is chosen to be used in the coefficient matrices, to prove the stability, then all \( A^\tilde{\mu}_i \)'s are independent of \( i \). Thus, all \( G^\tilde{\mu}_i \)'s and \( S^\tilde{\mu}_i \)'s are independent of \( i \) also. In this case, we can also arrive at the same inequalities as in (4.11) except that \( \gamma \) in Remark 1 is equal to 0.

A similar proof can be used for the 2D Log-Elastographic central difference scheme. To do this, we define

\[
G^\tilde{\mu}_i(\xi) = \frac{1}{2} \left( I - \frac{\Delta x}{\Delta y} A^\tilde{\mu}_i \right) e^{k\xi} + \frac{1}{2} \left( I + \frac{\Delta x}{\Delta y} A^\tilde{\mu}_i \right) e^{-k\xi}.
\]

Applying again the Von Neumann stability analysis but now to the following non-linear discretized scheme

\[
W_{i+1,j} = \frac{1}{2} (W_{i,j+1} + W_{i,j-1}) - \frac{\Delta x}{2\Delta y} A^\tilde{\mu}_i (W_{i,j+1} - W_{i,j-1}) - \Delta x E^\tilde{\mu}_i,
\]
where $W_{i,j}$ is the approximated values of $w = (\ln \tilde{\mu}, \tilde{p})^T$, we arrive at

$$\hat{W}_{i+1}(\xi) = G_{\tilde{\mu}}(\xi) \hat{W}_i(\xi) - \Delta x E_{\tilde{\mu}} e^{-k_j \xi}.$$  

Following the same steps as in the proof of Theorem 4.3 we can derive that

$$\| \prod_{m=l}^{n} \hat{G}_{\tilde{\mu}}(\xi) \|_2 \leq \bar{C}_1 \bar{C}_2 e^{\gamma(n-l)\Delta x}$$

for any $l$, where $\bar{C}_1$, $\bar{C}_2$ and $\gamma$ are defined as in Theorem 4.3. This leads to the following theorem.

**Theorem 4.4:** Consider the following 2D Log-Elastographic central difference scheme

$$\tilde{\mu}_{i+1,j} = \left( \frac{\tilde{\mu}_{i,j+1} - \Delta x E_{\tilde{\mu}}}{2\Delta y} \tilde{\mu}_{i,j-1} \right) \cdot \exp \left( -\Delta x \left( E_{\tilde{\mu}}(1,1) + E_{\tilde{\mu}}(1,2) - E_{\tilde{\mu}}(2,1) \tilde{\mu}_{i,j} \right) \right)$$

$$\tilde{p}_{i+1,j} = \frac{1}{2} (\tilde{p}_{i,j+1} + \tilde{p}_{i,j-1}) - \frac{\Delta x}{2\Delta y} \left( E_{\tilde{\mu}}(2,1) \tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1} \right)$$

and assume: (1) the eigenvalues $\lambda^1$, $\lambda^2$ of the matrix $E^1$ are real; (2) the step sizes $\Delta x$ and $\Delta y$ satisfy $(\Delta x/\Delta y) \max(|\lambda^1|, |\lambda^2|) \leq 1$; (3) the scaled shear modulus $\tilde{\mu}_{i,j}$ has a lower bound $L_{\mu}$, an upper bound $U_{\mu}$ and the numerical solution $\tilde{\mu}_{i,j}$ has a lower bound $L_{\mu}/2$, an upper bound $2U_{\mu}$ for all $i$ and $j$; (4) the initial values $\tilde{\mu}_0$ are real and satisfy $|\tilde{\mu}_0| \geq 1$; and (5) the total variation $TV(\tilde{\mu})$ is bounded. Then the discretized solution $\tilde{\mu}_{i,j}$ and $\tilde{p}_{i,j}$ satisfy

$$\prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \leq e^{\sqrt{2c} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2c}}}, \quad \|\tilde{\mu}_{n+1}\|_2 \leq c \|\ln \tilde{\mu}_0\|_2 + \bar{c},$$

where $c$ and $\bar{c}$ are the same constants as defined in Theorem 4.3.


4.3 Stability Analysis for the 2D Log-Elastographic Methods for the Elastic System with the $\epsilon\mu \nabla p$ Term

In this section, we follow a similar procedure as in Section 4.2 to establish the stability for the 2D Log-Elastographic algorithms for the elastic system with the $\epsilon\mu \nabla p$ term.

**Theorem 4.5:** Consider the following 2D Log-Elastographic upwind scheme

$$\tilde{\mu}_{i+1,j} = \left( \frac{\nabla x}{\Delta y} \right) (E^{1,+}(1,2)\tilde{\mu}_{i,j} - \tilde{\mu}_{i,j-1}) + E^{1,-}(1,2)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j}) (1 + \epsilon \tilde{\mu}_{i,j}) \right) \cdot \exp \left( -\Delta x (E^2(1,1) + E^3(1)/\tilde{\mu}_{i,j}) \right) \right) \right) \right) \right) \right) \right) \right)$$

and assume: (1) the eigenvalues $\lambda^1$, $\lambda^2$ of the matrix $E^1$ are real; (2) the step sizes $\Delta x$ and $\Delta y$ satisfy $(\Delta x/\Delta y) \max(|\lambda^1|, |\lambda^2|) \leq 1$; (3) the scaled shear modulus $\tilde{\mu}_{i,j}$ has a lower bound $L_\mu$, an upper bound $U_\mu$ and the numerical solution $\tilde{\mu}_{i,j}$ has a lower bound $L_\mu/2$, an upper bound $2U_\mu$ for all $i$ and $j$; (4) the initial values $\tilde{\mu}_0$ are real and satisfy $|\tilde{\mu}_0| \geq 1$; and (5) the total variation $TV(\tilde{\mu})$ is bounded. Then the discretized solution $\tilde{\mu}_{i,j}$ and $\tilde{\mu}_{i,j}$ satisfy

$$\prod_{j=1}^N \tilde{\mu}_{n+1,j}^{1/N} \leq e^{\sqrt{2c}/N} (\prod_{j=1}^N \tilde{\mu}_{0,j})^{\sqrt{2c}/N}, \quad \|\tilde{\mu}_{n+1}\|_2 \leq c_\epsilon \ln \tilde{\mu}_0 + \bar{c}_\epsilon, \quad (4.19)$$

where $c_\epsilon$ and $\bar{c}_\epsilon$ are constants.

**Remark 1:** The constants $c_\epsilon$ and $\bar{c}_\epsilon$ can be estimated in terms of the coefficient matrices. If we define $C_1$, $C_2$ to satisfy $\|S\|_2 \leq C_1$, $\|S^{-1}\|_2 \leq C_2$, where $S$ is the
eigenvector matrix of $E^1$, then

$$c_\epsilon = \max((\bar{C}_1 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\bar{\mu}}^2|) + \bar{c}\epsilon^2)$$

$$\cdot (\bar{C}_2 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\bar{\mu}}^2|) + \bar{c}\epsilon^2), 1)e^\gamma,$$

$$\bar{c}_\epsilon = \max((\bar{C}_1 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\bar{\mu}}^2|) + \bar{c}\epsilon^2)$$

$$\cdot (\bar{C}_2 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\bar{\mu}}^2|) + \bar{c}\epsilon^2), 1)(n + 1)\Delta xe^\gamma$$

$$\cdot (\bar{C}^2 + 2\epsilon|E^2(2, 1)\bar{\mu} + |E^3(2)||\bar{\mu})^{1/2},$$

where $\bar{C}_i = \max(2/L\tilde{\mu}, 2U_{\bar{\mu}})C_i$, $i = 1, 2$, $\bar{c} = 8|S(2, 2)U_{\bar{\mu}}^3|$, $\bar{\gamma} = TV(\tilde{\mu})(\frac{\|D_m\|_2}{4L\tilde{\mu}} + \epsilon|S(1, 1)S(2, 2)||\bar{\mu}^2| + |\tilde{c}|^2)/|d|$, $d$ is the determinant of the matrix $S$, $|\tilde{c}| = |S(1, 1)S(2, 2)|(U_{\bar{\mu}}/L\tilde{\mu})^2$ and $\bar{C}$ is defined as in Theorem 4.3.

**Remark 2:** This theorem establishes the stability for a discretized equation system (4.17) and (4.18). These two discretized equations and the two discretized equations (3.33) and (3.32) given in Chapter 3, and used for all our computations when we include the nonlinear term, are similar but not the same. They differ by the choice of the value of $\tilde{\mu}$ in the term $1 + \epsilon\tilde{\mu}$ in the discretized equation for $\tilde{p}$. The reason for this is that, while we can prove stability for (4.17)-(4.18) we have observed that our recoveries are improved when we replace (4.18) by (3.32). Note that this difference between the discretized model used to establish stability and the discretized model used for computation does not occur in the case where the nonlinear term is neglected.

**Proof.** As in the proof of Theorem 4.3, we first establish a stability result for the upwind scheme we obtain prior to our exponentiation step. Again we freeze the value of $\tilde{\mu}$ to be $\tilde{\mu}^x$ for each fixed $x$ and where the $\tilde{\mu}$ occurs in the coefficient matrices. When we do this, the discretized system before taking the exponential becomes

$$W_{i+1,j} = W_{i,j} - \frac{\Delta x}{\Delta y} \left( \bar{A}_{i,j}^x (W_{i,j} - W_{i,j-1}) + \bar{A}_{i,j}^- (W_{i,j+1} - W_{i,j}) \right) - \Delta x \bar{E}_{i,j}^x$$

(4.20)
where again $\mathbf{W}_{i,j}$ is the approximated value of $\mathbf{w} = (\ln \tilde{\mu}, \tilde{p})^T$ at $(i\Delta x, j\Delta y)$, while

$$
\bar{\tilde{A}}_{\tilde{\mu}}^x = \begin{pmatrix}
E^1(1,1) & E^1(1,2)(1 + \epsilon \tilde{\mu}_i^x)/\tilde{\mu}_i^x \\
E^1(2,1)\tilde{\mu}_i^x/(1 + \epsilon \tilde{\mu}_i^x) & E^1(2,2)
\end{pmatrix},
$$

$$
\bar{\tilde{E}}_{\tilde{\mu}}^x = \begin{pmatrix}
E^2(1,1) + E^3(1)/\tilde{\mu}_i^x \\
(E^2(2,1)\tilde{\mu}_i^x + E^3(2))/(1 + \epsilon \tilde{\mu}_i^x)
\end{pmatrix}.
$$

Apply the Von Neumann stability analysis and let

$$
\hat{\mathbf{W}}_{i+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-kj\xi} \mathbf{W}_{i+1,j}
$$

where $k = \sqrt{-1}$. Then we can obtain

$$
\hat{\mathbf{W}}_{i+1}(\xi) = \tilde{G}^x_{\tilde{\mu}}(\xi) \hat{\mathbf{W}}_i(\xi) - \Delta x \bar{\mathbf{E}}_{\tilde{\mu}}^x e^{-kj\xi}
$$

where

$$
\tilde{G}^x_{\tilde{\mu}}(\xi) = \left( I - \frac{\Delta x}{\Delta y} \bar{\tilde{A}}_{\tilde{\mu}}^x, + \frac{\Delta x}{\Delta y} \bar{\tilde{A}}_{\tilde{\mu}}^x, - \right)
+ \frac{\Delta x}{\Delta y} \bar{\tilde{A}}_{\tilde{\mu}}^x, + e^{-k\xi} - \frac{\Delta x}{\Delta y} \bar{\tilde{A}}_{\tilde{\mu}}^x, - e^{k\xi}.
$$

After $n + 1$ iterations,

$$
\hat{\mathbf{W}}_{n+1} = \tilde{G}^x_{\tilde{\mu}}(\xi) \hat{\mathbf{W}}_n - \Delta x \bar{\mathbf{E}}_{\tilde{\mu}}^x e^{-kj\xi}
= \tilde{G}^x_{\tilde{\mu}}(\xi) \left( \tilde{G}^x_{\tilde{\mu}}(\xi) \hat{\mathbf{W}}_{n-1} - \Delta x \bar{\mathbf{E}}_{\tilde{\mu}}^x e^{-kj\xi} \right) - \Delta x \bar{\mathbf{E}}_{\tilde{\mu}}^x e^{-kj\xi}
= \tilde{G}^x_{\tilde{\mu}}(\xi) \left( \tilde{G}^x_{\tilde{\mu}}(\xi) \hat{\mathbf{W}}_{n-2} - \Delta x \bar{\mathbf{E}}_{\tilde{\mu}}^x e^{-kj\xi} \right)
- \Delta x \left( \tilde{G}^x_{\tilde{\mu}}(\xi) \bar{\mathbf{E}}_{\tilde{\mu}}^x + \bar{\mathbf{E}}_{\tilde{\mu}}^x \right) e^{-kj\xi}
= \cdots
= \prod_{m=0}^{n} \tilde{G}^x_{\tilde{\mu}}(\xi) \hat{\mathbf{W}}_0 - \Delta x \left( \prod_{m=1}^{n} \tilde{G}^x_{\tilde{\mu}}(\xi) \bar{\mathbf{E}}_{\tilde{\mu}}^x + \sum_{m=2}^{n} \tilde{G}^x_{\tilde{\mu}}(\xi) \tilde{\mathbf{E}}_{\tilde{\mu}}^x + \cdots \right) e^{-kj\xi},
$$
implying that
\[ \|\tilde{W}_{n+1}\|_2 \leq \prod_{m=0}^{n} \|\tilde{G}^{\tilde{\mu}^x_m}(\xi)\|_2 \|\tilde{W}_0\|_2 + \Delta x \left( \prod_{m=1}^{n} \|\tilde{G}^{\tilde{\mu}^x_m}(\xi)\|_2 + \prod_{m=2}^{n} \|\tilde{G}^{\tilde{\mu}^x_m}(\xi)\|_2 
+ \cdots + \|\tilde{G}^{\tilde{\mu}^x_n}\|_2 + 1 \right) \max_m \|\tilde{E}^{\tilde{\mu}^x_m}\|_2. \]

In order to establish the stability results, we need to find non-negative constants \( \bar{K} \) and \( \bar{\alpha} \) such that
\[ \|\prod_{m=l}^{n} \tilde{G}^{\tilde{\mu}^x_m}(\xi)\|_2 \leq \bar{K} e^{\bar{\alpha}(n-l+1)\Delta x} \]
for any \( l \).

From Chapter 3 we know that the eigenvector matrix \( \tilde{S}^{\tilde{\mu}^x_i} \) of \( \tilde{A}^{\tilde{\mu}^x_i} \) and its inverse matrix \( (\tilde{S}^{\tilde{\mu}^x_i})^{-1} \) are given by
\[
\tilde{S}^{\tilde{\mu}^x_i} = \begin{pmatrix}
S(1, 1)(1 + \epsilon \tilde{\mu}^x_i) / \tilde{\mu}^x_i & S(1, 2) \\
S(2, 1) & S(2, 2) \tilde{\mu}^x_i / (1 + \epsilon \tilde{\mu}^x_i)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
S(1, 1)(1 + \epsilon \tilde{\mu}^x_i) / \tilde{\mu}^x_i & S(1, 2) \\
S(2, 1) & S(2, 2) \tilde{\mu}^x_i \sum_{n=0}^{\infty} (-\epsilon \tilde{\mu}^x_i)^n
\end{pmatrix}
\]
\[
= \tilde{S}^{\tilde{\mu}^x_i} + \epsilon \begin{pmatrix}
S(1, 1) & 0 \\
0 & S(2, 2) \tilde{\mu}^x_i \sum_{n=1}^{\infty} (-\epsilon \tilde{\mu}^x_i)^n \epsilon^{n-1}
\end{pmatrix}
\]
\[
(S^{\tilde{\mu}^x_i})^{-1} = \frac{1}{\bar{d}} \begin{pmatrix}
S(2, 2) \tilde{\mu}^x_i (1 + \epsilon \tilde{\mu}^x_i) / \tilde{\mu}^x_i & -S(1, 2) \\
-S(2, 1) & S(1, 1)(1 + \epsilon \tilde{\mu}^x_i) / \tilde{\mu}^x_i
\end{pmatrix}
\]
\[
= \frac{1}{\bar{d}} \begin{pmatrix}
S(2, 2) \tilde{\mu}^x_i \sum_{n=0}^{\infty} (-\epsilon \tilde{\mu}^x_i)^n & -S(1, 2) \\
-S(2, 1) & S(1, 1)(1 + \epsilon \tilde{\mu}^x_i) / \tilde{\mu}^x_i
\end{pmatrix}
\]
\[
= (S^{\tilde{\mu}^x_i})^{-1} + \frac{\epsilon}{\bar{d}} \begin{pmatrix}
S(2, 2) \tilde{\mu}^x_i \sum_{n=1}^{\infty} (-\tilde{\mu}^x_i)^n \epsilon^{n-1} & 0 \\
0 & S(1, 1)
\end{pmatrix}
\]
where as before \( S(i, j) \) is the component of the eigenvector matrix \( S \) of \( E^1 \) at the
\( i \)th row and the \( j \)th column, and \( d \) is the determinant of \( S \). And hence,

\[
\| \tilde{S}\tilde{\mu}_i \|_2 \leq \| S\tilde{\mu}_i \|_2 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{1\mu}^2|) + \tilde{c}\epsilon^2,
\]

\[
\| (\tilde{S}\tilde{\mu}_i)^{-1} \|_2 \leq \| (S\tilde{\mu}_i)^{-1} \|_2 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{1\mu}^2|)/|d| + \tilde{c}\epsilon^2/|d|,
\]

where \( \tilde{c} = 8|S(2, 2)U_{1\mu}^3| \) is a constant. In addition,

\[
\tilde{\mu}_m^x (\tilde{S}\tilde{\mu}_{m+1}^x)^{-1}
\]

\[
= \frac{1}{d} \begin{pmatrix}
S(1, 1)(1 + e\tilde{\mu}_m^x)\tilde{\mu}_m^x/S(1, 2) & S(2, 1) \\
S(2, 2)\tilde{\mu}_{m+1}^x/(1 + e\tilde{\mu}_m^x) & S(2, 2)\tilde{\mu}_m^x/(1 + e\tilde{\mu}_m^x)
\end{pmatrix}
\]

\[
= \frac{1}{d} \begin{pmatrix}
S(1, 1)S(2, 2)(1+e\tilde{\mu}_m^x)\tilde{\mu}_{m+1}^x & S(1, 2) - S(1, 2)(1+e\tilde{\mu}_m^x) \\
S(2, 1)S(2, 2)(1+e\tilde{\mu}_m^x) & S(1, 1)(1+e\tilde{\mu}_m^x)/\tilde{\mu}_{m+1}^x - S(1, 2)(1+e\tilde{\mu}_m^x)
\end{pmatrix}
\]

\[
= I + \frac{1}{d} \begin{pmatrix}
S(1, 1)S(2, 2)(1+e\tilde{\mu}_m^x)\tilde{\mu}_{m+1}^x & S(1, 2) - S(1, 2)(1+e\tilde{\mu}_m^x) \\
S(2, 1)S(2, 2)(1+e\tilde{\mu}_m^x) & S(1, 1)(1+e\tilde{\mu}_m^x)/\tilde{\mu}_{m+1}^x - S(1, 2)(1+e\tilde{\mu}_m^x)
\end{pmatrix}
\]

\[
= I + \frac{\tilde{\mu}_m^x - \tilde{\mu}_{m+1}^x}{d} \tilde{D}_m
\]

for any \( m \), where

\[
\tilde{D}_m = \begin{pmatrix}
\frac{S(1, 1)S(2, 2)}{\tilde{\mu}_m^x(1+e\tilde{\mu}_m^x)} & -S(1, 1)S(2, 1) \\
S(2, 1)S(2, 2)(1+e\tilde{\mu}_m^x) & S(1, 1)(1+e\tilde{\mu}_m^x)/\tilde{\mu}_{m+1}^x - S(1, 2)(1+e\tilde{\mu}_m^x)
\end{pmatrix}
\]

\[
= \frac{1}{\tilde{\mu}_m^x(1+e\tilde{\mu}_m^x)} D_m
\]

\[
+ \epsilon \begin{pmatrix}
-S(1, 1)S(2, 2)\tilde{\mu}_{m+1}^x/\tilde{\mu}_m^x & 0 \\
-S(2, 1)S(2, 2)(\tilde{\mu}_m^x + \tilde{\mu}_{m+1}^x) & -S(1, 1)S(2, 2)\tilde{\mu}_{m+1}^x/\tilde{\mu}_m^x
\end{pmatrix}
\]

\[
+ \frac{\tilde{c}\epsilon^2}{d}
\]

where \( \tilde{c} \) is a matrix with

\[
\tilde{c}(1, 1) = S(1, 1)S(2, 2) \sum_{n=2}^{\infty} ((-\tilde{\mu}_{m+1}^x)^n \epsilon^{n-2})/\tilde{\mu}_m^x, \quad \tilde{c}(1, 2) = 0,
\]
\[ \tilde{c}(2, 1) = S(2, 1)S(2, 2) \sum_{n=1}^{\infty} ((-\tilde{\mu}_m^x)^n \epsilon^{n-1}) \sum_{l=1}^{\infty} ((-\tilde{\mu}_{m+1}^x)^l \epsilon^{l-1}), \]

\[ \tilde{c}(2, 2) = S(1, 1)S(2, 2) \sum_{n=2}^{\infty} ((-\tilde{\mu}_m^x)^n \epsilon^{n-2})/\tilde{\mu}_{m+1}^x. \]

Then the following bound can be established:

\[ \| \tilde{D}_m \|_2 \leq \frac{\| D_m \|_2}{4|\tilde{L}_\mu|^2} + \epsilon |S(1, 1)S(2, 2)| \frac{|U_{\tilde{\mu}}|}{\tilde{L}_\mu} + |\tilde{c}| \epsilon^2 \]

where |\( \tilde{c} \)| = |S(1, 1)S(2, 2)|(\( U_{\tilde{\mu}} / \tilde{L}_\mu \))^2. Therefore,

\[
\prod_{m=l}^{n-1} \| S_{\tilde{\mu}_m^x} (S_{\tilde{\mu}_{m+1}^x})^{-1} \|_2 \leq \prod_{m=1}^{n-1} \| I + \frac{\tilde{\mu}_m^x - \tilde{\mu}_{m+1}^x}{d} \tilde{D}_m \|_2 \\
\leq \prod_{m=1}^{n-1} \left( 1 + \left\| \frac{\tilde{\mu}_m^x - \tilde{\mu}_{m+1}^x}{d} \tilde{D}_m \right\|_2 \right) \\
\leq \prod_{m=1}^{n-1} e^{\sum_{m=1}^{n-1} \left\| \frac{\tilde{\mu}_m^x - \tilde{\mu}_{m+1}^x}{d} \tilde{D}_m \right\|_2} \\
\leq e^{\sum_{m=1}^{n-1} \left\| \frac{\tilde{\mu}_m^x - \tilde{\mu}_{m+1}^x}{d} \tilde{D}_m \right\|_2} \leq e^{\gamma} 
\]

where \( \gamma = TV(\tilde{\mu}) \left( \frac{\| D_m \|_2}{4|\tilde{L}_\mu|^2} + \epsilon |S(1, 1)S(2, 2)| \frac{|U_{\tilde{\mu}}|}{\tilde{L}_\mu} + |\tilde{c}| \epsilon^2 \right) / |d| \).

As in the proof of Theorem 4.3, multiplying \( \tilde{G}_{\tilde{\mu}_l^x} \) on the left by \( \tilde{S}_{\tilde{\mu}_l^x} \) and on the right by \((\tilde{S}_{\tilde{\mu}_l^x})^{-1}\), we get

\[ H = \tilde{S}_{\tilde{\mu}_l^x} \tilde{G}_{\tilde{\mu}_l^x} (\tilde{S}_{\tilde{\mu}_l^x})^{-1} = I - \frac{\Delta x}{\Delta y} \left( \Lambda^+(1 - e^{-k \xi}) + \Lambda^-(e^{k \xi} - 1) \right). \]

From Lemma 4.1, we know that if \( \Delta x/\Delta y \max(|\lambda^1|, |\lambda^2|) \leq 1 \), then \( \| H \|_2 \leq 1 \), and hence \( \| H \|_2^{n-l+1} \leq 1 \) for any \( l \). Therefore,

\[
\| \prod_{m=l}^{n} \tilde{G}_{\tilde{\mu}_m^x}(\xi) \|_2 = \| \prod_{m=l}^{n} ( (\tilde{S}_{\tilde{\mu}_m^x})^{-1} H \tilde{S}_{\tilde{\mu}_m^x}) \|_2 \\
\leq \| (\tilde{S}_{\tilde{\mu}_l^x})^{-1} \|_2 \| \tilde{S}_{\tilde{\mu}_l^x} \|_2 \prod_{m=l}^{n-1} \| \tilde{S}_{\tilde{\mu}_m^x} (\tilde{S}_{\tilde{\mu}_{m+1}^x})^{-1} \|_2. 
\]
And hence, $\|\tilde{W}_{n+1}\|_2 \leq c_\epsilon \|\tilde{W}_0\|_2 + \tilde{c}_\epsilon$, where

$$c_\epsilon = \max((\tilde{C}_1 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\tilde{\mu}}^2|) + \tilde{c}\epsilon^2)$$

$$\cdot(C_2 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\tilde{\mu}}^2|) + \tilde{c}\epsilon^2), 1)e^{\tilde{c}^2},$$

$$\tilde{c}_\epsilon = \max((\tilde{C}_1 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\tilde{\mu}}^2|) + \tilde{c}\epsilon^2)$$

$$\cdot(C_2 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\tilde{\mu}}^2|) + \tilde{c}\epsilon^2), 1)(n + 1)\Delta x e^{\tilde{c}^2}$$

$$\cdot(\tilde{c}^2 + 2\epsilon(\tilde{C}_1 + \epsilon \max(|S(1, 1)|, 4|S(2, 2)U_{\tilde{\mu}}^2|) + \tilde{c}\epsilon^2))^{1/2}.$$

Since we set the initial value $\tilde{\mu}_0$ to be zero, this means that

$$\|\ln \tilde{\mu}_{n+1}\|_2 \leq c_\epsilon \|\ln \tilde{\mu}_0\|_2 + \tilde{c}_\epsilon, \quad \|\tilde{\mu}_{n+1}\|_2 \leq c_\epsilon \|\ln \tilde{\mu}_0\|_2 + \tilde{c}_\epsilon.$$

Lastly, we show that $\tilde{\mu}_{n+1,j}$ can also be bounded. From

$$\ln |a| \leq |\ln a| \Rightarrow |a| \leq e^{\ln a}$$

for any complex number $a$, and

$$\sum_{j=1}^N |\ln \tilde{\mu}_{n+1,j}| \leq \sqrt{2}\|\ln \tilde{\mu}_{n+1}\|_2 \leq \sqrt{2}(c_\epsilon \|\ln \tilde{\mu}_0\|_2 + \tilde{c}_\epsilon),$$

we can derive that

$$\prod_{j=1}^N |\tilde{\mu}_{n+1,j}| \leq \prod_{j=1}^N e^{\ln \tilde{\mu}_{n+1,j}} = e^{\sum_{j=1}^N |\ln \tilde{\mu}_{n+1,j}|}$$

$$\leq e^{\sqrt{2}c_\epsilon} \cdot e^{\sqrt{2}\epsilon \|\ln \tilde{\mu}_0\|_2}.$$

So if $\tilde{\mu}_{0,j}$ are real and satisfy $\tilde{\mu}_{0,j} \geq 1$ for all $j$, then

$$\prod_{j=1}^N |\tilde{\mu}_{n+1,j}| \leq e^{\sqrt{2}c_\epsilon} \left(\prod_{j=1}^N \tilde{\mu}_{0,j}\right)^{\sqrt{2}c_\epsilon}.$$
Taking the $N$th root of both sides we obtain that the geometric mean of $\tilde{\mu}_{n+1}$ satisfies
\[
\left( \prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \right)^{1/N} \leq e^{\sqrt{2}c_\epsilon/N} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2}c_\epsilon/N}.
\]

This finishes our proof. \qed

A similar proof can also be used for the 2D Log-Elastographic central difference scheme to obtain the following theorem.

**Theorem 4.6:** Consider the following 2D Log-Elastographic central difference scheme
\[
\tilde{\mu}_{i+1,j} = \left( \frac{1}{2} \left( \tilde{\mu}_{i,j+1} + \tilde{\mu}_{i,j-1} \right), \frac{1}{2} \left( \tilde{\mu}_{i+1,j} + \tilde{\mu}_{i-1,j} \right) \right.
\]
\[
- \frac{\Delta x}{2 \Delta y} \left( (1 - \tilde{\mu}_{i,j})(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) \right)
\]
\[
\cdot \exp \left( - \Delta x \left( E^1(1, 1) + E^3(1) \tilde{\mu}_{i,j} \right) \right)
\]
\[
\tilde{p}_{i+1,j} = \frac{1}{2} \left( \tilde{p}_{i,j+1} + \tilde{p}_{i,j-1} \right) - \frac{\Delta x}{2 \Delta y} \left( (1 + \tilde{\mu}_{i,j})(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1})/(1 + \tilde{\mu}_{i,j}) \right)
\]
\[
+ E^1(2, 2) \left( \tilde{p}_{i,j+1} - \tilde{p}_{i,j-1} \right) - \Delta x \left( E^2(2, 1) \tilde{\mu}_{i,j} + E^3(2) \right)/(1 + \tilde{\mu}_{i,j})
\]

and assume: (1) the eigenvalues $\lambda^1$, $\lambda^2$ of the matrix $E^1$ are real; (2) the step sizes $\Delta x$ and $\Delta y$ satisfy $(\Delta x/\Delta y) \max(|\lambda^1|, |\lambda^2|) \leq 1$; (3) the scaled shear modulus $\tilde{\mu}_{i,j}$ has a lower bound $L_\mu$, an upper bound $U_\mu$ and the numerical solution $\tilde{\mu}_{i,j}$ has a lower bound $L_\mu/2$, an upper bound $2U_\mu$ for all $i$ and $j$; (4) the initial values $\tilde{\mu}_0$ are real and satisfy $|\tilde{\mu}_0| \geq 1$; and (5) the total variation $TV(\tilde{\mu})$ is bounded. Then the discretized solution $\tilde{\mu}_{i,j}$ and $\tilde{p}_{i,j}$ satisfy
\[
\left( \prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \right)^{1/N} \leq e^{\sqrt{2}c_\epsilon/N} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2}c_\epsilon/N}, \quad \|\tilde{p}_{n+1}\|_2 \leq c_\epsilon \|\ln \tilde{\mu}_0\|_2 + \tilde{c}_\epsilon, \quad (4.21)
\]

where $c_\epsilon$ and $\tilde{c}_\epsilon$ are constants as defined in Theorem 4.5.
4.4 Order of Accuracy for the 2D Log-Elastographic Algorithms

In this section we establish the order of accuracy for the 2D Log-Elastographic algorithm. Theorem 4.8 shows that the 2D Log-Elastographic upwind scheme is of first order accuracy and Theorem 4.9 shows that the 2D Log-Elastographic central difference scheme is almost of first order accuracy.

**Theorem 4.8:** The 2D Log-Elastographic upwind scheme

\[
\tilde{\mu}_{i+1,j} = \left( \tilde{\mu}_{i,j} \right) \cdot \frac{\Delta y}{\Delta x} (E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1)) \cdot \frac{\Delta y}{\Delta x} (E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1)) \cdot \exp \left( -\Delta x E_{i,j}^{2}(1,1) - \Delta x E_{i,j}^{3}(1,1) \right)
\]

\[
\tilde{\mu}_{i-1,j} = \tilde{\mu}_{i,j+1} + \tilde{\mu}_{i,j-1}
\]

is of first order accuracy, if \(\omega_{xx}, (E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1))\omega_{yy}, (E_{i,j}^{1,-}(1,2) - E_{i,j}^{1,+}(1,2))q_{yy}, E_{i,j}^{1,+}(1,1)E_{i,j}^{1,-}(1,1)\omega_{xx}, (E_{i,j}^{2}(1,1) + E_{i,j}^{3}(1,1))\omega_{xx}, (E_{i,j}^{3}(1,1) + E_{i,j}^{3}(1,1))\omega_{yy}, q_{yy}\) are uniformly bounded throughout the whole computational domain, where \(U_{i,j} = \left( \begin{array}{c} \omega_{i,j} \\ q_{i,j} \end{array} \right)\) is the exact solution at the point \((x_i, y_j)\).
Proof. For the equation (4.22), the truncation error satisfies

\[
\Delta x \tau_{i,j} = \omega_{i+1,j} - \left( \omega_{i,j} \right) \left( 1 - \frac{\Delta x}{2} (E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1)) \right) \\
- \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(1,2) (q_{i,j} - q_{i,j-1}) + E_{i,j}^{1,-}(1,2) (q_{i,j+1} - q_{i,j}) \right) \\
\cdot \exp \left( - \frac{\Delta x E_{i,j}^{2}(1,1) + \Delta x E_{i,j}^{3}(1)/\omega_{i,j}}{} \right)
\]

where \( i_1, i_2, j_1, j_2, j_3 \) and \( j_4 \) are the appropriate points given to us from the Taylor series remainder term. Let

\[
\alpha := \left( \omega_{i,j} \right) \left( 1 - \frac{\Delta x}{2} (E_{i,j}^{1,+}(1,1) - E_{i,j}^{1,-}(1,1)) \right) \left( \omega_{i,j} \right)
\]

\[
\beta := \left( \omega_{i,j} \right) \left( 1 - \frac{\Delta x}{2} (E_{i,j}^{2}(1,1) - E_{i,j}^{3}(1)/\omega_{i,j}) \right) \left( \omega_{i,j} \right)
\]

then according to \((a + b)^r = \sum_{k=0}^{\infty} \binom{r}{k} a^{r-k} b^k\), for any complex \( r \) and when
\[ |a| \geq |b|, \]

\[
\alpha = \omega_{i,j} \frac{\Delta x}{\Delta y} E_{i,j}^{1,1}(1,1) + \frac{\Delta x}{\Delta y} E_{i,j}^{1,1}(1,1) \omega_{i,j} \cdot \left( - \Delta y (\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} \right)
\]

\[
+ \left( \frac{\Delta x}{\Delta y} E_{i,j}^{1,1}(1,1) \right) \omega_{i,j} \cdot \left( - \Delta y (\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} \right) \cdot \left( - \Delta y (\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} \right) + \cdots
\]

\[
\beta = \omega_{i,j} \frac{\Delta x}{\Delta y} E_{i,j}^{1,1}(1,1) - \frac{\Delta x}{\Delta y} E_{i,j}^{1,1}(1,1) \omega_{i,j} \cdot \left( - \Delta y (\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} \right)
\]

\[
+ \left( \frac{\Delta x}{\Delta y} E_{i,j}^{1,1}(1,1) \right) \omega_{i,j} \cdot \left( - \Delta y (\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} \right) \cdot \left( - \Delta y (\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} \right) + \cdots
\]

And hence,

\[
\Delta x \tau_{i,j} = \omega_{i,j} + \frac{\Delta x}{\Delta y} (\omega_y)_{i,j} + \frac{\Delta x^2}{2} (\omega_{xx})_{i,j} - \left( \omega_{i,j} - \Delta x \left( E_{i,j}^{1,1}(1,1) + E_{i,j}^{1,-1}(1,1) \right) \cdot \Delta y (\omega_y)_{i,j}
\]

\[
+ \frac{\Delta x}{\Delta y} \cdot \frac{\Delta y^2}{2} \left( E_{i,j}^{1,1}(1,1) (\omega_{yy})_{i,j} - E_{i,j}^{1,-1}(1,1) (\omega_{yy})_{i,j} \right)
\]

\[
+ \Delta x^2 E_{i,j}^{1,1}(1,1) \omega_{i,j} (\omega_y)_{i,j}^2 + O(\Delta x^2 \Delta y)
\]

\[
- \frac{\Delta x}{\Delta y} \left( \Delta y \left( E_{i,j}^{1,1}(1,1) + E_{i,j}^{1,-1}(1,1) \right) (q_y)_{i,j} \right)
\]

\[
- \frac{\Delta x}{\Delta y} \cdot \frac{\Delta y^2}{2} \left( E_{i,j}^{1,-1}(1,1) (q_{yy})_{i,j} - E_{i,j}^{1,1}(1,1) (q_{yy})_{i,j} \right)
\]

\[
\cdot \left( 1 - \Delta x \left( E_{i,j}^{2,1}(1,1) + E_{i,j}^{2,3}(1,1) \omega_{i,j} \right) + \frac{\Delta x^2}{2} \left( E_{i,j}^{2,1}(1,1) + E_{i,j}^{2,3}(1,1) \omega_{i,j} \right) \right)
\]

\[
= \Delta x \left( \omega_x + E_{i,j}^{1,1}(1,1) \omega_y + E_{i,j}^{1,2}(1,1) q_y + E_{i,j}^{2,1}(1,1) \omega + E_{i,j}^{3,1}(1,1) \right)
\]

\[
+ \frac{\Delta x^2}{2} (\omega_{xx})_{i,j} - \frac{\Delta x \Delta y}{2} \left( E_{i,j}^{1,1}(1,1) (\omega_{yy})_{i,j} - E_{i,j}^{1,-1}(1,1) (\omega_{yy})_{i,j} \right)
\]

\[
- \Delta x^2 E_{i,j}^{1,1}(1,1) E_{i,j}^{1,-1}(1,1) \omega_{i,j} (\omega_y)_{i,j}^2 + O(\Delta x^2 \Delta y)
\]

\[
+ \frac{\Delta x^2}{2} \left( E_{i,j}^{2,1}(1,1) (q_y)_{i,j} - E_{i,j}^{2,3}(1,1) (q_y)_{i,j} \right)
\]

\[
- \frac{\Delta x}{2} \left( E_{i,j}^{2,1}(1,1) + E_{i,j}^{2,3}(1,1) \omega_{i,j} \right)^2 \omega_{i,j}
\]

\[
+ \Delta x^2 \left( E_{i,j}^{2,1}(1,1) + E_{i,j}^{2,3}(1,1) \omega_{i,j} \right) \left( E_{i,j}^{1,1}(1,1) (\omega_y)_{i,j} + E_{i,j}^{2,1}(1,1) (q_y)_{i,j} \right).
\]
Since $\omega$ is the exact solution to the first equation,

$$(\omega_x + E_1^1(1, 1)\omega_y + E_1^1(1, 2)q_y + E_1^2(1, 1)\omega + E_1^3(1))_{i,j} = 0.$$  

And hence,

$$\tau_{i,j} = \frac{\Delta x}{2}(\omega_{xx})_{i,j} - \frac{\Delta y}{2}(E_{i,j}^{1+}(1, 1)(\omega_{yy})_{i,j1} - E_{i,j}^{1-}(1, 1)(\omega_{yy})_{i,j2})$$

$$- \Delta x E_{i,j}^{1+}(1, 1)E_{i,j}^{1-}(1, 1)\omega_{i,j}^{-1}(\omega_{yy})_{i,j}^2 + O(\Delta x \Delta y)$$

$$+ \frac{\Delta y}{2}(E_{i,j}^{1-}(1, 2)(q_{yy})_{i,j4} - E_{i,j}^{1+}(1, 2)(q_{yy})_{i,j3})$$

$$- \Delta x \left(\frac{3\omega_{i,j}}{2}(E_{i,j}^2(1, 1) + E_{i,j}^3(1)/\omega_{i,j})^2ight)$$

$$+ (\omega_x)_{i,j}\left(E_{i,j}^2(1, 1) + E_{i,j}^3(1)/\omega_{i,j}\right).$$  \hspace{1cm} (4.26)

Therefore, if $\omega_{xx}, (E_{i,j}^{1+}(1, 1) - E_{i,j}^{1-}(1, 1))\omega_{yy}, (E_{i,j}^{1+}(1, 2) - E_{i,j}^{1-}(1, 2))q_{yy},$ $E_{i,j}^{1+}(1, 1)E_{i,j}^{1-}(1, 1)\omega^{-1}\omega_y^2, (E_{i,j}^2(1, 1) + E_{i,j}^3(1)/\omega)\omega_x$ and $(E_{i,j}^2(1, 1) + E_{i,j}^3(1)/\omega)^2/\omega$ are uniformly bounded throughout the whole computational domain, then

$$||\tau_i||_\infty = O(\Delta x) + O(\Delta y) \rightarrow 0, \text{ as } \Delta x, \Delta y \rightarrow 0;$$

or if

$$\sum_j |(\omega_{xx})_{i,j}|^2 < \alpha_1 < \infty, \sum_j |(E_{i,j}^{1+}E_{i,j}^{1-}\omega^{-1}\omega_{yy})|^2 < \alpha_2 < \infty$$

$$\sum_j |((E_{i,j}^{-1}(1, 1) - E_{i,j}^{1+}(1, 1))\omega_{yy} + (E_{i,j}^{-1}(1, 2) - E_{i,j}^{1+}(1, 2))q_{yy})_{i,j}|^2 < \alpha_3 < \infty$$

$$\sum_j |((E_{i,j}^2(1, 1) + E_{i,j}^3(1)/\omega)\omega_x)_{i,j}|^2 < \alpha_4 < \infty$$

$$\sum_j |((E_{i,j}^2(1, 1) + E_{i,j}^3(1)/\omega)^2/\omega)_{i,j}|^2 < \alpha_5 < \infty$$

then $||\tau_i||_2 = O(\Delta x) + O(\Delta y) \rightarrow 0, \text{ as } \Delta x, \Delta y \rightarrow 0.$
Next, let’s look at the truncation error to the equation (4.23)

\[
\Delta x \bar{\tau}_{i,j} = q_{i+1,j} - q_{i,j} + \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(2,1)(\omega_{i,j} - \omega_{i,j-1}) + E_{i,j}^{1,-}(2,1)(\omega_{i,j+1} - \omega_{i,j}) \right. \\
+ E_{i,j}^{1,+}(2,2)(q_{i,j} - q_{i,j-1}) + E_{i,j}^{1,-}(2,2)(q_{i,j+1} - q_{i,j}) \left. \right) \\
+ \Delta x E_{i,j}^2(2,1)\omega_{i,j} + \Delta x E_{i,j}^3(2) \\
= q_{i,j} + \Delta x(q_x)_{i,j} + \frac{\Delta x^2}{2}(q_{xx})_{i,j} - q_{i,j} \\
+ \frac{\Delta x}{\Delta y} \left( E_{i,j}^{1,+}(2,1)(\omega_{i,j} - \omega_{i,j} + \Delta y(\omega_y)_{i,j} - \frac{\Delta y^2}{2}(\omega_{yy})_{i,j}) \right. \\
+ E_{i,j}^{1,-}(2,1)(\omega_{i,j} + \Delta y(\omega_y)_{i,j} + \frac{\Delta y^2}{2}(\omega_{yy})_{i,j} - \omega_{i,j}) \left. \right) \\
+ E_{i,j}^{1,+}(2,2)(q_{i,j} - q_{i,j} + \Delta y(q_y)_{i,j} - \frac{\Delta y^2}{2}(q_{yy})_{i,j}) \\
+ E_{i,j}^{1,-}(2,2)(q_{i,j} + \Delta y(q_y)_{i,j} + \frac{\Delta y^2}{2}(q_{yy})_{i,j} - q_{i,j}) \\
+ \Delta x E_{i,j}^2(2,1)\omega_{i,j} + \Delta x E_{i,j}^3(2) \\
= \Delta x \left( q_x + \left( E_{i,j}^{1,+}(2,1) + E_{i,j}^{1,-}(2,1) \right) \omega_y \right. \\
+ \left( E_{i,j}^{1,+}(2,2) + E_{i,j}^{1,-}(2,2) \right) q_y + E^2(2,1)\omega + E^3(2) \right)_{i,j} \\
+ \frac{\Delta x^2}{2}(q_{xx})_{i,j} + \frac{\Delta x \Delta y}{2} \left( E_{i,j}^{1,-}(2,1)(\omega_{yy})_{i,j} - E_{i,j}^{1,+}(2,1)(\omega_{yy})_{i,j} \right) \\
+ \frac{\Delta x \Delta y}{2} \left( E_{i,j}^{1,+}(2,2)(q_{yy})_{i,j} - E_{i,j}^{1,-}(2,2)(q_{yy})_{i,j} \right) \tag{4.27}
\]

where \(i_1, i_2, j_1, j_2, j_3\) and \(j_4\) are the appropriate points given to us from the Taylor series remainder term. Since \(q\) is the exact solution to the second equation,

\[
(q_x + E^1(2,1)\omega_y + E^1(2,2)q_y + E^2(2,1)\omega + E^3(2))_{i,j} = 0.
\]

And hence,

\[
\bar{\tau}_{i,j} = \frac{\Delta x}{2}(q_{xx})_{i,j} + \frac{\Delta y}{2} \left( E_{i,j}^{1,-}(2,1)(\omega_{yy})_{i,j} - E_{i,j}^{1,+}(2,1)(\omega_{yy})_{i,j} \right) \\
+ \frac{\Delta y}{2} \left( E_{i,j}^{1,+}(2,2)(q_{yy})_{i,j} - E_{i,j}^{1,-}(2,2)(q_{yy})_{i,j} \right). \tag{4.28}
\]

Therefore, if \(q_{xx}, (E_{i,j}^{1,-}(2,1) - E_{i,j}^{1,+}(2,1))\omega_{yy}, (E_{i,j}^{1,-}(2,2) - E_{i,j}^{1,+}(2,2))q_{yy}\) are un-
formly bounded throughout the whole computational domain, then

\[ \|\bar{\tau}_i\|_\infty = O(\Delta x) + O(\Delta y) \to 0, \quad \text{as } \Delta x, \Delta y \to 0; \]

or if

\[ \sum_j |(q_{xx})_{i,j}|^2 < \alpha_1 < \infty \]
\[ \sum_j |((E_{i,j}^{1,-}(2,1) - E_{i,j}^{1,+}(2,1))\omega_{yy})_{i,j}|^2 < \alpha_2 < \infty, \]
\[ \sum_j |(E_{i,j}^{1,-}(2,2) - E_{i,j}^{1,+}(2,2))q_{yy})_{i,j}|^2 < \alpha_3 < \infty \]

then \[ \|\bar{\tau}_i\|_2 = O(\Delta x) + O(\Delta y) \to 0, \quad \text{as } \Delta x, \Delta y \to 0. \]

The above analysis shows that the 2D Log-Elastographic upwind algorithm is of first order accuracy.

**Theorem 4.9:** The following 2D Log-Elastographic nonlinear central difference scheme

\[
\tilde{\mu}_{i+1,j} = \left( \frac{1}{2} \left( \mu_{i,j+1} + \mu_{i,j-1} \right) - \frac{\Delta x_i}{2\Delta y} E_{i,j}^1(2,1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) \right) \cdot \exp(-\Delta x_i(E_{i,j}^2(1,1) + E_{i,j}^3(1)/\mu_{i,j})) \tag{4.29}
\]

\[
\tilde{p}_{i+1,j} = \frac{1}{2} \left( \tilde{p}_{i,j+1} + \tilde{p}_{i,j-1} \right) - \frac{\Delta x_i}{2\Delta y} (E_{i,j}^1(2,1)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1}) + E_{i,j}^2(2,2)(\tilde{\mu}_{i,j+1} - \tilde{\mu}_{i,j-1})) - \Delta x_i E_{i,j}^2(2,1)\tilde{\mu}_{i,j} - \Delta x_i E_{i,j}^3(2) \tag{4.30}
\]

is almost of first order accuracy if \((E_{i,j}^2(1,1))^2\omega^{-1}(\omega_y)^2, (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega)\omega_x, \omega(E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega)^2, \omega_{xx}, \omega_{yy}, q_{xx} \text{ and } q_{yy}\) are uniformly bounded throughout the whole computational domain, where \(U_{i,j} = \begin{pmatrix} \omega \\ q \end{pmatrix}\) is the exact solution at point \((x_i, y_j)\).
Proof. For the equation (4.29), the truncation error satisfies

\[
\Delta x_{i} \tau_{i,j} = \omega_{i+1,j} - \left( \omega_{i,j+1} + \frac{\Delta x_{i}}{2\Delta y} E_{i,j}^{1,1,1} \right) \cdot \omega_{i,j-1} - \frac{\Delta x_{i}}{2\Delta y} E_{i,j}^{1,2}(1,2) (q_{i,j+1} - q_{i,j-1}) \cdot \exp \left( -\Delta x_{i} (E_{i,j}^{2,1,1} + E_{i,j}^{3,1,1}) / \omega_{i,j} \right)
\]

\[
= (\omega_{i,j} + \Delta x_{i} (\omega_{x})_{i,j} + \frac{\Delta x_{i}^{2}}{2} (\omega_{xx})_{i,j}) - \left( (\omega_{i,j} + \Delta y (\omega_{y})_{i,j} + \frac{\Delta y^{2}}{2} (\omega_{yy})_{i,j} + \frac{\Delta y^{3}}{6} (\omega_{yyy})_{i,j}) \omega_{i,j} \right) \frac{1}{2} \left( 1 - \frac{\Delta x_{i}}{\Delta y} E_{i,j}^{1,1,1} \right) \cdot \exp \left( -\Delta x_{i} (E_{i,j}^{2,1,1} + E_{i,j}^{3,1,1}) / \omega_{i,j} \right) - \Delta x_{i} E_{i,j}^{2,1,1}(1,2) \cdot \exp \left( -\Delta x_{i} (E_{i,j}^{2,1,1} + E_{i,j}^{3,1,1}) / \omega_{i,j} \right)
\]

(4.31)

where \( i_{1}, i_{2}, j_{1}, j_{2}, j_{3} \) and \( j_{4} \) are the appropriate points given to us from the Taylor series remainder term. Let

\[
\alpha := (\omega_{i,j} + \Delta y (\omega_{y})_{i,j} + \frac{\Delta y^{2}}{2} (\omega_{yy})_{i,j} + \frac{\Delta y^{3}}{6} (\omega_{yyy})_{i,j}) \omega_{i,j} \frac{1}{2} \left( 1 - \frac{\Delta x_{i}}{\Delta y} E_{i,j}^{1,1,1} \right)
\]

\[
\beta := (\omega_{i,j} - \Delta y (\omega_{y})_{i,j} + \frac{\Delta y^{2}}{2} (\omega_{yy})_{i,j} - \frac{\Delta y^{3}}{6} (\omega_{yyy})_{i,j}) \omega_{i,j} \frac{1}{2} \left( 1 + \frac{\Delta x_{i}}{\Delta y} E_{i,j}^{1,1,1} \right)
\]

then according to \((a + b)^{r} = \sum_{k=0}^{\infty} \binom{r}{k} a^{r-k} b^{k}\), for any complex \( r \) and when \(|a| \geq |b|\),

\[
\alpha = \omega_{i,j} \left( \frac{1}{2} \left( 1 - \frac{\Delta x_{i}}{\Delta y} E_{i,j}^{1,1,1} \right) + \frac{1}{2} \left( 1 - \frac{\Delta x_{i}}{\Delta y} E_{i,j}^{1,1,1} \right) \right) - \Delta x_{i} E_{i,j}^{2,1,1}(1,2) \omega_{i,j} \frac{1}{2} \left( 1 - \frac{\Delta x_{i}}{\Delta y} E_{i,j}^{1,1,1} \right) \cdot \exp \left( -\Delta x_{i} (E_{i,j}^{2,1,1} + E_{i,j}^{3,1,1}) / \omega_{i,j} \right)
\]

(4.32)
\[
\beta = \frac{1}{2} (1 + \frac{\Delta x_i}{\Delta y} E_{i,j}^1(1,1,1)) + \frac{1}{2} (1 + \frac{\Delta x_i}{\Delta y} E_{i,j}^1(1,1)) \omega_{i,j}^2 (\frac{\Delta x_i}{\Delta y} E_{i,j}^1(1,1,1) - 1)
\]

\[
\cdot (-\Delta y(\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j} - \frac{\Delta y^3}{6} (\omega_{yyy})_{i,j}^2)
\]

\[
+ \left( \frac{1}{2} (1 + \frac{\Delta x_i}{\Delta y} E_{i,j}^1(1,1)) \right) \omega_{i,j}^4 (\Delta x_i \omega_y)^2 + \cdot \cdot \cdot . \quad (4.33)
\]

And hence,

\[
\Delta x_i \tau_{i,j} = \omega_{i,j} + \Delta x_i (\omega_x)_{i,j} + \frac{\Delta x_i^2}{2} (\omega_{xx})_{i,j} - (\omega_{i,j} + \frac{1}{2} (1 - \frac{\Delta x_i}{\Delta y} E_{i,j}^1(1,1)) (\Delta y(\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j}^2)
\]

\[
+ \frac{1}{2} (1 + \frac{\Delta x_i}{\Delta y} E_{i,j}^1(1,1)) (-\Delta y(\omega_y)_{i,j} + \frac{\Delta y^2}{2} (\omega_{yy})_{i,j}^2)
\]

\[
+ \frac{1}{4} (1 - \frac{\Delta x_i}{\Delta y} (E_{i,j}^1(1,1))^2) (-\Delta y^2 (\omega_y)_{i,j}^2) + O(\Delta y^3) + O(\Delta x_i^2 \Delta y)
\]

\[
- \frac{\Delta x_i}{2 \Delta y} E_{i,j}^1(1,2) (2 \Delta y(q_y)_{i,j} + \frac{\Delta y^3}{6} ((q_{yyy})_{i,j} + (q_{yyy})_{i,j}))
\]

\[
\cdot (1 - \Delta x_i (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega_{i,j} + \frac{\Delta x_i^2}{2} (E_{i,j}^1(1,1) + E_{i,j}^3(1)/\omega_{i,j})^2))
\]

\[
= \Delta x_i (\omega_x + E_{i,j}^1(1,1) \omega_y + E_{i,j}^1(1,2) q_y + E_{i,j}^2(1,1) \omega + E_{i,j}^3(1))_{i,j}
\]

\[
+ \frac{\Delta x_i^2}{2} (\omega_{xx})_{i,j} + \frac{\Delta y^2}{2} ((\omega_{yy})_{i,j} + (\omega_{yy})_{i,j}^2)
\]

\[
- \frac{1}{4} (1 - \frac{\Delta x_i}{\Delta y} (E_{i,j}^1(1,1))^2) \omega_{i,j}^{-1} (\Delta y^2 (\omega_y)_{i,j}^2) + O(\Delta y^3) + O(\Delta x_i^2 \Delta y)
\]

\[
+ \frac{\Delta x_i}{12} (q_{yyy})_{i,j} + (q_{yyy})_{i,j}) - \frac{\Delta x_i}{2} (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega_{i,j})^2 \omega_{i,j}
\]

\[
+ \Delta x_i E_{i,j}^1(1,1) (\omega_y)_{i,j} (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega_{i,j}) + O(\Delta x_i \Delta y^2)
\]

\[
+ \Delta x_i E_{i,j}^1(1,2) (q_y)_{i,j} (E_{i,j}^1(1,1) + E_{i,j}^3(1)/\omega_{i,j}). \quad (4.34)
\]
Since $\omega$ is the exact solution to the first equation, we obtain

$$
\tau_{i,j} = \frac{\Delta x_i}{2} (\omega_{xx})_{i,j} - \frac{\Delta y^2}{2\Delta x_i} (\omega_{yy})_{i,j} + (\omega_{yy})_{i,j} + \frac{1}{2} \omega_i^{-1} (\omega_y)_{i,j}^2 \\
+ \frac{1}{4} \Delta x_i (E_{i,j}^1(1,1))^2 \omega_i^{-1} (\omega_y)_{i,j}^2 + \mathcal{O}\left(\frac{\Delta y^3}{\Delta x_i}\right) + \mathcal{O}(\Delta x_i \Delta y) \\
+ \frac{\Delta y^2}{12} (q_{y_{yy}})_{i,j} + (q_{y_{yy}})_{i,j} - \Delta x_i ((\omega_x)_{i,j} (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega_{i,j}) \\
+ \frac{3}{2} \omega_{i,j} (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega_{i,j})^2). \\
(4.35)
$$

If $\omega_{xx}, \omega_{yy}, (E_{i,j}^1(1,1))^2 \omega^{-1}(\omega_y)^2, (E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega) \omega_x, \omega(E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega)^2$ are uniformly bounded throughout the whole computational domain, then

$$
\|\bar{\tau}_i\|_2 = \mathcal{O}(\Delta x_i) + \mathcal{O}\left(\frac{\Delta y^2}{\Delta x_i}\right) \to 0, \text{ as } \Delta x_i, \Delta y \to 0;
$$
or if

$$
\sum_j |((E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega) \omega_x)_{i,j}|^2 < \alpha_1 < \infty, \\
\sum_j |(\omega_{xx})_{i,j}|^2 < \alpha_2 < \infty, \quad \sum_j |(\omega_{yy})_{i,j}|^2 < \alpha_3 < \infty, \\
\sum_j |((E_{i,j}^1(1,1))^2 \omega^{-1}(\omega_y)^2)_{i,j}|^2 < \alpha_4 < \infty \\
\sum_j |(\omega(E_{i,j}^2(1,1) + E_{i,j}^3(1)/\omega)^2)_{i,j}|^2 < \alpha_5 < \infty
$$
then

$$
\|\bar{\tau}_i\|_2 = \mathcal{O}(\Delta x_i) + \mathcal{O}\left(\frac{\Delta y^2}{\Delta x_i}\right) \to 0, \text{ as } \Delta x_i, \Delta y \to 0.
$$
Next, let us look at the truncation error to the equation (4.30),

\[ \Delta x_i \bar{\tau}_{i,j} = q_{i+1,j} - \frac{1}{2}(q_{i,j+1} + q_{i,j-1}) + \frac{\Delta x_i}{2\Delta y}(E_{i,j}^1(2,1)(\omega_{i,j+1} - \omega_{i,j-1}) + E_{i,j}^1(2,2)(q_{i,j+1} - q_{i,j-1})) + \Delta x_i E_{i,j}^2(2,1)\omega_{i,j} + \Delta x_i E_{i,j}^3(2) \]

\[ = q_{i,j} + \Delta x_i(q_{xx})_{i,j} + \frac{\Delta x_i^2}{2}(q_{xx})_{i,j} - \frac{1}{2}((q_{i,j} + \Delta y(q_y)_{i,j} + \frac{\Delta y^2}{2}(q_{yy})_{i,j}) + (q_{i,j} + \Delta y(q_y)_{i,j} + \frac{\Delta y^2}{2}(q_{yy})_{i,j})) \]

\[ + \frac{\Delta x_i}{2\Delta y}(E_{i,j}^1(2,1)(\omega_{i,j} + \Delta y(\omega_y)_{i,j} + \frac{\Delta y^2}{2}(\omega_{yy})_{i,j} + \frac{\Delta y^3}{6}(\omega_{yyyy})_{i,j,j}) - \omega_{i,j} + \Delta y(\omega_y)_{i,j} - \frac{\Delta y^2}{2}(\omega_{yy})_{i,j} + \frac{\Delta y^3}{6}(\omega_{yyyy})_{i,j,j}) \]

\[ + E_{i,j}^1(2,2)(q_{i,j} + \Delta y(q_y)_{i,j} + \frac{\Delta y^2}{2}(q_{yy})_{i,j} + \frac{\Delta y^3}{6}(q_{yyyy})_{i,j,j}) - q_{i,j} + \Delta y(q_y)_{i,j} - \frac{\Delta y^2}{2}(q_{yy})_{i,j} + \frac{\Delta y^3}{6}(q_{yyyy})_{i,j,j}) \]

\[ + \Delta x_i E_{i,j}^2(2,1)\omega_{i,j} + \Delta x_i E_{i,j}^3(2) \]

\[ = \Delta x_i(q_x + (E_{i,j}^1+(2,1) + E_{i,j}^{1+}(2,1))\omega_y + (E_{i,j}^{1+}(2,2) + E_{i,j}^{1+}(2,2))q_y + E_{i,j}^2(2,1)\omega + E_{i,j}^3(2))_{i,j} + \frac{\Delta x_i^2}{2}(q_{xx})_{i,j} - \frac{\Delta y^2}{4}((q_{yy})_{i,j1} + (q_{yy})_{i,j2}) \]

\[ + \frac{\Delta x_i \Delta y^2}{12}E_{i,j}^1(2,1)((\omega_{yyy})_{i,j3} + (\omega_{yyyy})_{i,j4}) + E_{i,j}^1(2,2)((q_{yy})_{i,j5} + (q_{yyyy})_{i,j6}) \] \hspace{1cm} (4.36)

where \( i_1, i_2, j_1, j_2, j_3, j_4, j_5 \) and \( j_6 \) are the appropriate points given to us from the Taylor series remainder term. Since \( \omega \) is the exact solution to the first equation, we obtain

\[ \bar{\tau}_{i,j} = \frac{\Delta x_i}{2}(q_{xx})_{i,j} - \frac{\Delta y^2}{4\Delta x_i}((q_{yy})_{i,j1} + (q_{yy})_{i,j2}) \]

\[ + \frac{\Delta y^2}{12}E_{i,j}^1(2,1)((\omega_{yyy})_{i,j3} + (\omega_{yyyy})_{i,j4}) + E_{i,j}^1(2,2)((q_{yy})_{i,j5} + (q_{yyyy})_{i,j6}) \] \hspace{1cm} (4.37)

If \( q_{xx} \) and \( q_{yy} \) are both uniformly bounded throughout the whole computational
domain, then $\|\vec{r}_i\|_\infty = O(\Delta x_i) + O(\frac{\Delta y^2}{\Delta x_i}) \to 0$, as $\Delta x_i, \Delta y \to 0$; or if
\[
\sum_j \|(q_{xx})_{i,j}\|^2 < \alpha_1 < \infty, \quad \sum_j \|(q_{yy})_{i,j}\|^2 < \alpha_2 < \infty
\]

then $\|\vec{r}_i\|_2 = O(\Delta x_i) + O(\frac{\Delta y^2}{\Delta x_i}) \to 0$, as $\Delta x_i, \Delta y \to 0$.

Therefore, the 2D Log-Elastographic nonlinear central difference scheme is almost of first order accuracy.
CHAPTER 5
Forward Simulation and Numerical Examples

In the previous chapters we introduced the 2D Log-Elastographic nonlinear finite difference algorithms to reconstruct the scaled shear modulus and the pressure simultaneously using a 2D plane strain elastic system. In this chapter, we test these algorithms with some numerical examples using synthetic data. In order to obtain the synthetic data and to apply our 2D Log-Elastographic algorithms, we need to do the forward simulation, i.e., to solve the 2D plane strain elastic system for the displacement (here we solve it also for the pressure). We start this chapter by presenting a finite difference solver together with a perfectly matched layer method for the forward simulation, and then we give some computational results for testing our 2D Log-Elastographic algorithms. This is applied when $\tilde{\mu} \in C^2$.

5.1 Forward Simulation

First we need to simulate wave propagation described by the following 2D plane strain elastic system

$$u_{tt} = \nabla \cdot (\tilde{\mu}(\nabla u + \nabla u^T)) + \nabla \tilde{p} + \tilde{f} \quad (5.1)$$

together with

$$\nabla \cdot u = \epsilon \tilde{p} \quad (5.2)$$

where we assume that the density $\rho$ is a constant, and hence $\tilde{\mu} = \mu/\rho$, $\tilde{p} = p/\rho$, $\tilde{f} = f/\rho$, and $\epsilon = \rho/\lambda$ is assumed to be constant, see (1.3). Here, our object is to solve the above system for the displacement $u$ and the pressure $\tilde{p}$ with the scaled shear modulus $\tilde{\mu}$ and $\epsilon$ known. Furthermore $u$ is the displacement of a propagating wave and we will be simulating an outgoing wave. To avoid artificial reflections from the boundary of our computational domain, we will rewrite our system (5.1)-(5.2) as a hyperbolic system, first in $u$, $\tilde{p}$ and later for the variables $u$, $q$ where
\[ q = \tilde{p} + 2\nabla \mu \cdot \mathbf{u} / (1 + \epsilon \tilde{\mu}), \]
and then apply a Perfectly Matched Layer (PML) method in the neighborhood of the boundary. The reason for changing to the variables \( \mathbf{u}, q \) is that the formulas for the PML method are more easily stated.

Note also that as we mentioned before \( \nabla \cdot \mathbf{u} \) and \( \epsilon \) are both very small quantities. This is another reason that we cannot solve the equation (5.2) directly for \( \tilde{p} \), because by doing this small amount of noise in the measured displacement \( \mathbf{u} \) can be amplified severely. To avoid this error amplification and also to accomplish the Perfectly Matched Layer idea, we first take the divergence of the equation system (5.1); then we arrive at a third equation which is hyperbolic in \( \tilde{p} \). This equation will replace (5.2) and will be coupled with the system (5.1).

\[
\epsilon \tilde{p}_{tt} = \nabla \cdot (\nabla \cdot (\tilde{\mu}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T))) + \Delta \tilde{p} + \nabla \cdot \tilde{f} \\
= \nabla \cdot (\tilde{\mu} \Delta \mathbf{u} + \nabla \tilde{p} \cdot \nabla \mathbf{u} + \tilde{\mu} \nabla \cdot (\nabla \mathbf{u})^T + \nabla \tilde{\mu} \cdot (\nabla \mathbf{u})^T) + \Delta \tilde{p} + \nabla \cdot \tilde{f} \\
= \nabla \tilde{\mu} \cdot \Delta \mathbf{u} + \epsilon \tilde{\mu} \Delta \tilde{p} + \nabla^2 \tilde{\mu} : (\nabla \mathbf{u})^T + \epsilon \nabla \tilde{\mu} \cdot \nabla \tilde{p} + \nabla \tilde{\mu} \cdot (\nabla \cdot (\nabla \mathbf{u})^T) \\
+ \tilde{\mu} \nabla \cdot (\nabla \cdot (\nabla \mathbf{u})^T) + \nabla^2 \tilde{\mu} : \nabla \mathbf{u} + \nabla \tilde{\mu} \cdot \Delta \mathbf{u} + \Delta \tilde{p} + \nabla \cdot \tilde{f} \\
= (1 + 2\epsilon \tilde{\mu}) \Delta \tilde{p} + 2\nabla \tilde{\mu} \cdot \Delta \mathbf{u} + 2\epsilon \nabla \tilde{\mu} \cdot \nabla \tilde{p} \\
+ \nabla^2 \tilde{\mu} : (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \nabla \cdot \tilde{f} \tag{5.3}
\]

where note that \( \nabla \cdot (\nabla \cdot (\nabla \mathbf{u})^T) = \epsilon \Delta \tilde{p} \) and \( \nabla \cdot (\nabla \cdot (\mathbf{u})^T) = \epsilon \nabla \tilde{p} \). Our goal is to apply a second order finite difference method to solve the system (5.1) and equation (5.3) together for the displacement \( \mathbf{u} \) and the pressure \( \tilde{p} \). While it turns out that the variables \( \mathbf{u}, \tilde{p} \) are not the best ones to accomplish this task.

Since we are simulating experiments implemented in a much larger region, we assume outgoing boundary conditions on all boundaries of our domain. In order to achieve this outgoing boundary condition, we apply the method of Perfectly Matched Layers, which was originally proposed by Berenger in [3]. Its basic idea is to set up absorbing layers (PML) around the boundaries of the computational domain such that waves incident upon the PML from a non-PML medium do not reflect at the interface. Our specific implementation follows Collino and Tsogka [13].

To obtain the absorbing layer implementation more easily, we will make a change of dependent variables and let \( q = \tilde{p} + \frac{2\nabla \tilde{\mu} \cdot \mathbf{u}}{1 + \epsilon \tilde{\mu}} \), then after some straightforward
calculation we obtain the equation system for $u, q$ as

$$
\begin{align*}
\mathbf{u}_{tt} &= \tilde{\mu} \Delta \mathbf{u} + \nabla \tilde{\mu} \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + (1 + \epsilon \tilde{\mu}) \nabla (q - \frac{2\nabla \tilde{\mu} \cdot \mathbf{u}}{1 + \epsilon \tilde{\mu}}) + \mathbf{f} \\
&= \tilde{\mu} \Delta \mathbf{u} + \nabla \tilde{\mu} \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + (1 + \epsilon \tilde{\mu}) \nabla q \\
&\quad -2(1 + \epsilon \tilde{\mu}) \frac{1}{1 + \epsilon \tilde{\mu}} (\nabla^2 \tilde{\mu} \cdot \mathbf{u} + \nabla \tilde{\mu} \cdot (\nabla \mathbf{u})^T) \\
&\quad - \frac{\epsilon}{(1 + \epsilon \tilde{\mu})^2} (\nabla \tilde{\mu} \cdot (\nabla \tilde{\mu})^T) \cdot \mathbf{u} + \mathbf{f}
\end{align*}
$$

$$
\begin{align*}
\mathbf{u}_{tt} &= \tilde{\mu} \Delta \mathbf{u} + \nabla \tilde{\mu} \cdot (\nabla \mathbf{u} - (\nabla \mathbf{u})^T) + (1 + \epsilon \tilde{\mu}) \nabla q \\
&\quad -2\nabla^2 \tilde{\mu} \cdot \mathbf{u} + \frac{2\epsilon}{1 + \epsilon \tilde{\mu}} (\nabla \tilde{\mu} \cdot (\nabla \tilde{\mu})^T) \cdot \mathbf{u} + \mathbf{f}
\end{align*}
$$

(5.4)

and

$$
\begin{align*}
\epsilon q_{tt} &= \epsilon \tilde{\mu}_{tt} + \frac{2\epsilon}{1 + \epsilon \tilde{\mu}} \nabla \tilde{\mu} \cdot \mathbf{u}_{tt} \\
&= (1 + 2\epsilon \tilde{\mu}) \Delta q - 2(1 + 2\epsilon \tilde{\mu}) \Delta \left( \frac{\nabla \tilde{\mu} \cdot \mathbf{u}}{1 + \epsilon \tilde{\mu}} \right) + 2\nabla \tilde{\mu} \cdot \Delta \mathbf{u} \\
&\quad + \nabla^2 \tilde{\mu} : (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + 2\epsilon \nabla \tilde{\mu} \cdot \nabla q - 4\epsilon \nabla \tilde{\mu} \cdot \nabla \left( \frac{\nabla \tilde{\mu} \cdot \mathbf{u}}{1 + \epsilon \tilde{\mu}} \right) + \nabla \cdot \mathbf{f} \\
&\quad + \frac{2\epsilon}{1 + \epsilon \tilde{\mu}} \nabla \tilde{\mu} \cdot (\tilde{\mu} \Delta \mathbf{u} + \nabla \tilde{\mu} \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + (1 + \epsilon \tilde{\mu}) \nabla q \\
&\quad -2(1 + \epsilon \tilde{\mu}) \nabla \left( \frac{2\nabla \tilde{\mu} \cdot \mathbf{u}}{1 + \epsilon \tilde{\mu}} \right) + \mathbf{f})
\end{align*}
$$

(5.5)
That is,
\[
q_{tt} = \frac{1 + 2\epsilon\tilde{\mu}}{\epsilon} \Delta q - \frac{2(1 + 2\epsilon\tilde{\mu})}{\epsilon(1 + \epsilon\tilde{\mu})} \nabla (\nabla^2 \tilde{\mu}) \cdot \mathbf{u} - \frac{2(1 + 3\epsilon\tilde{\mu})}{\epsilon(1 + \epsilon\tilde{\mu})} \nabla^2 \tilde{\mu} : (\nabla \tilde{\mu})^T
\]
\[
+ \frac{2(1 + 2\epsilon\tilde{\mu})}{(1 + \epsilon\tilde{\mu})^2} \Delta \tilde{\mu} (\nabla \tilde{\mu} \cdot \mathbf{u}) + \frac{4\epsilon\tilde{\mu}}{(1 + \epsilon\tilde{\mu})^2} \nabla \tilde{\mu} \cdot (\nabla \tilde{\mu} \cdot \nabla \mathbf{u})
\]
\[
- \frac{4}{(1 + \epsilon\tilde{\mu})^3} \nabla \tilde{\mu} \cdot (\nabla^2 \tilde{\mu} \cdot \mathbf{u}) + \frac{4\epsilon}{(1 + \epsilon\tilde{\mu})^3} \nabla \tilde{\mu} \cdot ((\nabla \tilde{\mu} \cdot (\nabla \tilde{\mu})^T) \cdot \mathbf{u})
\]
\[
+ 4\nabla \tilde{\mu} \cdot \nabla q + \frac{1}{\epsilon} \nabla \cdot \mathbf{f} + \frac{2}{1 + \epsilon\tilde{\mu}} \nabla \tilde{\mu} \cdot \mathbf{f}.
\]

What we have achieved is an equation system (5.4), (5.6) of the form

\[
\begin{pmatrix}
\mathbf{u} \\
q
\end{pmatrix}_{tt} = D \Delta \begin{pmatrix}
\mathbf{u} \\
q
\end{pmatrix} + \text{lower order derivative terms},
\]

where \( D \) is a diagonal matrix. Now it is straightforward to develop the first order p.d.e.s needed for our PML formulation, see [13].

In the layers, for displacement, we start by rewriting \( \mathbf{u} = \mathbf{u}^1 + \mathbf{u}^2 + \mathbf{u}^3 \), where \( \mathbf{u}^1 \), \( \mathbf{u}^2 \) and \( \mathbf{u}^3 \) satisfy the following first order p.d.e. systems,

\[
\mathbf{u}^1_t = \nabla \cdot \mathbf{V}
\]
\[
\mathbf{V}_t = \tilde{\mu} \nabla \mathbf{u} \tag{5.7}
\]
\[
\mathbf{u}^2_t = \mathbf{h}
\]
\[
h_t = \nabla \tilde{\mu} \cdot (\nabla \mathbf{u})^T \tag{5.8}
\]
\[
\mathbf{u}^3_t = (1 + \epsilon\tilde{\mu}) \nabla w
\]
\[
w_t = q - \frac{2\nabla \tilde{\mu} \cdot \mathbf{u}}{1 + \epsilon\tilde{\mu}} \tag{5.9}
\]

where we have introduced three new variables \( \mathbf{V} \), \( \mathbf{h} \) and \( w \). For our variable \( q \), we rewrite \( q = q^1 + q^2 \), where \( q_1 \) and \( q_2 \) satisfy the following p.d.e. system,

\[
q^1_t = \frac{1 + 2\epsilon\tilde{\mu}}{\epsilon} \nabla \cdot \mathbf{r}
\]
\[
\mathbf{r}_t = \nabla q \tag{5.10}
\]
\[
\begin{align*}
q_t^2 &= s \\
s_t &= -\frac{2(1 + 3\epsilon\tilde{\mu})}{\epsilon(1 + \epsilon\tilde{\mu})}\nabla^2\tilde{\mu} : (\nabla\tilde{\mu})^T \\
&\quad+ \frac{2(1 + 2\epsilon\tilde{\mu})}{(1 + \epsilon\tilde{\mu})^2}\Delta\tilde{\mu}(\nabla\tilde{\mu} \cdot \mathbf{u}) + \frac{4\epsilon\tilde{\mu}}{(1 + \epsilon\tilde{\mu})^2}\nabla\tilde{\mu} \cdot (\nabla\tilde{\mu} \cdot \nabla\mathbf{u}) \\
&\quad- \frac{4}{(1 + \epsilon\tilde{\mu})^2}\nabla\tilde{\mu} \cdot (\nabla^2\tilde{\mu} \cdot \mathbf{u}) + \frac{4\epsilon}{(1 + \epsilon\tilde{\mu})^3}\nabla\tilde{\mu} \cdot (\nabla\tilde{\mu} \cdot (\nabla\tilde{\mu})^T) \cdot \mathbf{u} \\
&\quad+ 4\nabla\tilde{\mu} \cdot \nabla q
\end{align*}
\]

where we have introduced two new variables \( r \) and \( s \).

Instead of solving these exact first order systems exactly in the layers, we will add terms that will damp out any waves that propagate to the layers. To do this, we apply the procedure introduced in [13]. First we decompose the solution \( \nu = \nu^x + \nu^y \), where \( \nu \) stands for any of the solutions \( u^1, u^2, u^3, q^1 \) or \( q^2 \), superscript \( x \) means that we keep only the \( x \) derivative, and superscript \( y \) means that we keep only the \( y \) derivative. Then, we introduce a function \( d(x) \), which is positive in the layers (1), (2), (3), (7), (8), (9) and zero otherwise, and a function \( d(y) \), which is positive in the layers (1), (4), (7), (3), (6), (9) and zero otherwise, see Figure 5.1. These functions play the role of damping factors, and are given in 2D by the following formula, where \( \delta \) is the width of the layer, \( R \) is the reflection coefficient and \( v \) is the wave velocity. Specifically, \( v = \sqrt{\mu} \) for \( u \) and \( v = \sqrt{(1 + 2\epsilon\tilde{\mu})/\epsilon} \) for \( q \).

\[
d(x) = \log \left( \frac{1}{R} \right) \frac{3v}{2\delta} \left( \frac{x}{\delta} \right)^2, \quad d(y) = \log \left( \frac{1}{R} \right) \frac{3v}{2\delta} \left( \frac{y}{\delta} \right)^2.
\]

To describe how the damping term is added, let us first remark that if an equation of the above first order systems is expressed as

\[
\nu_t = f(w_x) + g(w_y),
\]

where \( f \) and \( g \) are known linear functions and \( w \) is the dependent variable on the right hand side of the equation, and if no additional terms to achieve damping are
Figure 5.1: The different PML layers and the interior domain: $d(x)$ is positive in the layers (1), (2), (3), (7), (8), (9) and zero otherwise; $d(y)$ is positive in the layers (1), (4), (7), (3), (6), (9) and zero otherwise.

added then in the layer we would solve the following set of equations,

$$\nu_t^x = f(w_x),$$
$$\nu_t^y = g(w_y).$$

So to include the damping factor so that we damp out reflections we instead solve

$$\nu_t^x + d(x)\nu^x = f(w_x)$$
$$\nu_t^y + d(y)\nu^y = g(w_y)$$

in the layers.

We will utilize these equations and compute the solutions to the forward problem. In the forward simulation, we solve the system (5.1) and equation (5.3) inside the interior region, and solve a specific set of first order p.d.e. systems including the damping terms, inside the surrounding layers. The interior domain is $50 \times 50$ mm with $100 \times 100$ grid lines, and each of the surrounding layers has width of 7.5 mm with 15 grid lines.

Furthermore, for the spatial discretization, we apply a staggered marker-and-
cell (MAC) grid to obtain better stability for computing the pressure. The MAC grid method, introduced by Harlow and Welsh in [25], discretizes the region of consideration spatially into many small cells. Each cell has a pressure, $\bar{p}$, defined at its center. It also has a displacement, $\mathbf{u}$, but the components of the displacement are placed on the middle of the cell edges: $u_1$ on the two edges parallel to the $y$ axis and $u_2$ on the two edges parallel to the $x$ axis, see Figure 5.2. In such a way, we can avoid having two uncoupled networks of points where the pressure is calculated, thereby avoiding oscillations in the computed pressure. In our case where the medium is nearly incompressible, we have very small $\epsilon$, or equivalently very large $\lambda/\rho$, e.g. $\lambda/\rho \approx 10^6$. Correspondingly, the compression wave speed is very large. So in this case we apply an implicit finite difference method, the Crank-Nicolson scheme, to solve the equation (5.6) for $q$ in order to utilize a larger time step and save computational time.

![Figure 5.2: The MAC grid where ▲ stands for $u_1$, ■ stand for $u_2$ and ◦ stand for $q$.](image)

We use this forward algorithm to compute synthetic data for a selection of shear moduli, which are each represented by a $C^2$ function. Our inverse algorithms will be applied to this synthetic data and compared to the exact values.
5.2 Numerical Examples

In this section, we show some recoveries obtained by solving the 2D plane strain elastic system (3.3) for $\tilde{\mu}$, $\tilde{p}$ with our 2D Log-Elastographic algorithms introduced in Chapter 3 using synthetic data with and without noise. First we compare the recovered scaled shear moduli obtained from neglecting the nonlinear term, see Section 3.1, and the recovered scaled shear moduli obtained from including the nonlinear term, see Section 3.2; Second we compare the recovered shear moduli obtained from the 2D Log-Elastographic algorithms with (1) the recovered scaled shear moduli obtained from solving the equation (1.5) with the Direct Inversion method (I); (2) the recovered scaled shear moduli obtained from solving the acoustic equation (1.6) with the acoustic Log-Elastographic algorithm; and (3) the recovered scaled shear moduli obtained from solving the 2D plane strain elastic system for $\tilde{\mu}$ and $\tilde{p}$ with the standard upwind and central difference methods; Last we test the first order convergence results that we obtained in Chapter 4 with some numerical examples.

In all these examples, $\lambda/\rho = 10^6 m^2/s^2$; the central frequency for simulation and inversion is 100Hz; the units that we are using to display the results are seconds for time, meters for length and $m^2/s^2$ for the scaled shear modulus; the computational domain is

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 0.05, 0 \leq y \leq 0.05\};$$

and for the discretization, we use a $100 \times 100$ spatial grid. We add a source in the $y$ direction, that is, the second component of the source function $\tilde{f}$, which is given by

$$f_2(x, y, t) = 5000e^{-10^7(x-x_0)^2} e^{-t^2/2\sigma^2} \sin(\omega(t + \pi/2)),$$

where $\sigma = 0.005$, $\omega = 200\pi$ and $x_0 = 0.0005$. The rapid decay of the Gaussian functions makes it possible to simulate a short duration wave pulse temporally, with central frequency 100Hz, and to simulate a line source spatially.

Recall that we assume in the experiment that the wave propagation direction occurs primarily in one direction, say the $x$ direction. And following our algorithmic development for recovering $\tilde{\mu}$, $\tilde{p}$, we assume that the $x$ direction is a time-like direc-
tion. If the $y$ direction is the horizontal direction, then, in the 2D Log-Elastographic algorithm, we require a Dirichlet boundary condition on the top of the computational domain. We also need a Dirichlet condition on the left and right boundaries of the computational domain when the inflow condition, i.e., when the inflow occurs, holds. When we need boundary values, we choose the boundary values to be the results obtained from applying the Direct Inversion method (I) to equation (1.5) where $\hat{u}$ is the component of $u$ that is orthogonal to the wave propagation direction.

5.2.1 Recoveries from Neglecting the Nonlinear Term and from Including the Nonlinear Term

First we compare the recoveries obtained by solving the system (3.3) without the nonlinear term $\epsilon\mu\nabla p$ with the 2D Log-Elastographic algorithms and the recoveries obtained by solving the system (3.1) including the nonlinear term with the 2D Log-Elastographic algorithms. In the examples given below, there is one round inclusion and one elliptic inclusion inside the computational domain. The scaled shear modulus of the background is $2 \text{ m}^2/\text{s}^2$. The maximum scaled shear modulus inside the inclusions is $4 \text{ m}^2/\text{s}^2$, $8 \text{ m}^2/\text{s}^2$ and $12 \text{ m}^2/\text{s}^2$ respectively. Figures 5.3, 5.4 and 5.5 show the recoveries and the error between recoveries obtained by neglecting the nonlinear term and including the nonlinear term. From these images, you can see that the recoveries obtained from neglecting the nonlinear term and including the nonlinear term are very similar, and the error is of order $10^{-7}$ when the contrast ratio is 2, $10^{-6}$ when the contrast is 4, and $10^{-5}$ when the contrast is 6. For all these calculations $\lambda/\rho$ is $10^6 \text{ m}^2/\text{s}^2$. While we do not show it here we observe that these orders of errors are also true when we add 10% and 20% noise into the synthetic data. In our remaining examples, we thus ignore this nonlinear term.

5.2.2 Recoveries with Inversion Algorithms with No Noise in the Data

Next we show recoveries from synthetic data without noise. In all sets of figures given below, the first one is the exact scaled shear modulus; the second is the recovered scaled shear modulus obtained from the Direct Inversion method (I) where the locally constant assumption is made; the third is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\vec{\mu} \nabla \hat{u}^T)$ terms but including
Figure 5.3: (a) is the recovery from the 2D Log-Elastographic upwind scheme without the nonlinear term; (b) is the recovery from the 2D Log-Elastographic upwind scheme with the nonlinear term; (c) is the difference between recoveries obtained by neglecting the nonlinear term and including the nonlinear term with the 2D Log-Elastographic upwind scheme; (d) is the recovery from the 2D Log-Elastographic central difference scheme without the nonlinear term; (e) is the recovery from the 2D Log-Elastographic central difference scheme with the nonlinear term; (f) is the difference between recoveries obtained by neglecting the nonlinear term and including the nonlinear term with the 2D Log-Elastographic central difference scheme. The exact contrast between the background and the maximum value inside the inclusions is 2 to 4.

Note that in all cases, we have to differentiate the data. Since there is no noise, we use standard finite difference approximations to calculate the derivatives of the solution \( \hat{\mathbf{u}} \); those derivatives become coefficients in our inverse problem model.

(I) Example (1-2): in this set of examples, the scaled shear modulus \( \tilde{\mu} \) is given...
Figure 5.4: (a) is the recovery from the 2D Log-Elastographic upwind scheme without the nonlinear term; (b) is the recovery from the 2D Log-Elastographic upwind scheme with the nonlinear term; (c) is the difference between recoveries obtained by neglecting the nonlinear term and including the nonlinear term with the 2D Log-Elastographic upwind scheme; (d) is the recovery from the 2D Log-Elastographic central difference scheme without the nonlinear term; (e) is the recovery from the 2D Log-Elastographic central difference scheme with the nonlinear term; (f) is the difference between recoveries obtained by neglecting the nonlinear term and including the nonlinear term with the 2D Log-Elastographic central difference scheme. The exact contrast between the background and the maximum value inside the inclusions is 2 to 8.

by the following function:

\[
\tilde{\mu}(x, y) = b + a \ast ((r - (x - x_0)^2 - (y - y_0)^2)/r)^3
\]

where \( r = 0.006, b = 2, x_0 = 0.03, y_0 = 0.04 \) in both examples, \( a = 4 \) in the first example, and \( a = 16 \) in the second example. So there is one round inclusion inside the computational domain. The scaled shear modulus of the background is 2 m\(^2\)/s\(^2\) in both examples. The maximum scaled shear modulus inside the inclusion is 6 m\(^2\)/s\(^2\) in the first example and 18 m\(^2\)/s\(^2\) in the second example. Figure 5.6 and Figure 5.7 show the recoveries.

(II) Example (3-4): in this set of examples, the scaled shear modulus \( \tilde{\mu} \) is given
Figure 5.5: (a) is the recovery from the 2D Log-Elastographic upwind scheme without the nonlinear term; (b) is the recovery from the 2D Log-Elastographic upwind scheme with the nonlinear term; (c) is the difference between recoveries obtained by neglecting the nonlinear term and including the nonlinear term with the 2D Log-Elastographic upwind scheme; (d) is the recovery from the 2D Log-Elastographic central difference scheme without the nonlinear term; (e) is the recovery from the 2D Log-Elastographic central difference scheme with the nonlinear term; (f) is the difference between recoveries obtained by neglecting the nonlinear term and including the nonlinear term with the 2D Log-Elastographic central difference scheme. The exact contrast between the background and the maximum value inside the inclusions is 2 to 12.

by the following function:

\[
\tilde{\mu}(x, y) = b + a \ast \left(\frac{(r - (x - x_0)^2 - (y - y_0)^2)}{r}\right)^3 \\
+ a \ast \left(1 - \frac{(x + y)/\sqrt{2} - x_1)^2}{r_1} - \frac{(y - x)/\sqrt{2} - y_1)^2}{r_2}\right)^3
\]

where \(r = 0.003, r_1 = 0.005, r_2 = 0.009, b = 2, x_0 = 0.015, y_0 = 0.035, x_1 = 0.035, y_1 = -0.015\) in both examples, \(a = 2\) in the first example, and \(a = 10\) in the second example. So there is one round inclusion and one elliptic inclusion inside the computational domain. The scaled shear modulus of the background is \(2 \, m^2/s^2\) in both examples. The maximum scaled shear modulus inside the inclusions is \(4 \, m^2/s^2\) in the first example and \(12 \, m^2/s^2\) in the second example. Figure 5.8 and Figure 5.9 show the recoveries.
Figure 5.6: For example I(1) where the contrast between the background and the maximum value in the inclusion is 2 to 6: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and \( \nabla \cdot (\tilde{\nu} \nabla \hat{u}^T) \) terms but including the remaining \( \nabla \tilde{\mu} \) terms; (d) is the recovered shear modulus obtained by neglecting the \( \nabla \cdot (\tilde{\mu} \nabla \hat{u}^T) \) term but including the pressure and the remaining \( \nabla \tilde{\mu} \) terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic central difference scheme.
Figure 5.7: For example I(2) where the contrast between the background and the maximum value in the inclusion is 2 to 18: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ terms but including the remaining $\nabla \tilde{\mu}$ terms; (d) is the recovered shear modulus obtained by neglecting the $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ term but including the pressure and the remaining $\nabla \tilde{\mu}$ terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic central difference scheme.
For example II(3) where the contrast between the background and the maximum value in the inclusion is 2 to 4: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and \( \nabla \cdot (\vec{\mu} \nabla \hat{u}) \) terms but including the remaining \( \nabla \vec{\mu} \) terms; (d) is the recovered shear modulus obtained by neglecting the \( \nabla \cdot (\vec{\mu} \nabla \hat{u}) \) term but including the pressure and the remaining \( \nabla \vec{\mu} \) terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic central difference scheme.

Figure 5.8:
Figure 5.9: For example II(4) where the contrast between the background and the maximum value in the inclusion is 2 to 12: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\tilde{\mu} \nabla \tilde{u})$ terms but including the remaining $\nabla \tilde{\mu}$ terms; (d) is the recovered shear modulus obtained by neglecting the $\nabla \cdot (\tilde{\mu} \nabla \tilde{u})$ term but including the pressure and the remaining $\nabla \tilde{\mu}$ terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic central difference scheme.
(III) Example (5-6): in this set of examples, the scaled shear modulus \( \tilde{\mu} \) is

\[
\tilde{\mu}(x, y) = b + a \star ((r - (x - x_0)^2 - (y - y_0)^2)/r)^3 \\
+ a \star (1 - ((x + y)/\sqrt{2} - x_1)^2/r_1 - ((y - x)/\sqrt{2} - y_1)^2/r_2)^3 \\
+ a \star (1 - ((x - y)/\sqrt{2} - x_2)^2/r_1 - ((x + y)/\sqrt{2} - y_2)^2/r_2)^3
\]

where \( r = 0.003, r_1 = 0.005, r_2 = 0.009, b = 2, x_0 = 0.027, y_0 = 0.025, x_1 = 0.035, \\
y_1 = -0.017, x_2 = -0.015, y_2 = 0.037 \) in both examples, \( a = 2 \) in the first example, \\
and \( a = 10 \) in the second example. So there are two elliptic inclusions and one round inclusion inside the computational domain. The scaled shear modulus of the background is 2 m²/s² in both examples. The maximum scaled shear modulus inside the inclusions is 4 m²/s² in the first example and 12 m²/s² in the second example. Figure 5.10 and Figure 5.11 show the recoveries.

(IV) Example (7-8): in this set of examples, the scaled shear modulus \( \tilde{\mu} \) is

\[
\tilde{\mu}(x, y) = b + a \star (1 - ((x + y)/\sqrt{2} - x_1)^2/r_1 - ((y - x)/\sqrt{2} - y_1)^2/r_2)^3 \\
+ a \star (1 - ((x - y)/\sqrt{2} - x_2)^2/r_1 - ((x + y)/\sqrt{2} - y_2)^2/r_2)^3
\]

where \( r = 0.003, r_1 = 0.005, r_2 = 0.009, b = 2, x_1 = 0.015, y_1 = 0.033, x_2 = 0.02, \\
y_2 = -0.003 \) in both examples, \( a = 2 \) in the first example, and \( a = 14 \) in the second example. So there are two elliptic inclusions placed one in front of the other with respect to the time like \( x \) direction in the computational domain. The scaled shear modulus of the background is 2 m²/s² in both examples. The maximum scaled shear modulus inside the inclusions is 4 m²/s² in the first example and 16 m²/s² in the second example. Figure 5.12 and Figure 5.13 show the recoveries.

Comparing all these images, you can see that the maximum value of \( \tilde{\mu} \) in the recoveries from the Direct Inversion method (I) and from the acoustic solver exhibit undershooting, and the shape and size of the inclusions are not recovered well. In some cases, the amplitude of the small inclusion is significantly reduced. The
Figure 5.10: For example III(5) where the contrast between the background and the maximum value in the inclusion is 2 to 4: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\tilde{\mu} \nabla \tilde{u}^T)$ terms but including the remaining $\nabla \tilde{\mu}$ terms; (d) is the recovered shear modulus obtained by neglecting the $\nabla \cdot (\tilde{\mu} \nabla \tilde{u}^T)$ term but including the pressure and the remaining $\nabla \tilde{\mu}$ terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic upwind scheme, and (h) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic central difference scheme.
Figure 5.11: For example III(6) where the contrast between the background and the maximum value in the inclusion is 2 to 12: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\widetilde{\mu} \nabla \hat{u})$ terms but including the remaining $\nabla \widetilde{\mu}$ terms; (d) is the recovered shear modulus obtained by neglecting the $\nabla \cdot (\widetilde{\mu} \nabla \hat{u})$ term but including the pressure and the remaining $\nabla \widetilde{\mu}$ terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic central difference scheme.
Figure 5.12: For example IV(7) where the contrast between the background and the maximum value in the inclusion is 2 to 4: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\mu \nabla \hat{u}^T)$ terms but including the remaining $\nabla \mu$ terms; (d) is the recovered shear modulus obtained by neglecting the $\nabla \cdot (\mu \nabla \hat{u}^T)$ term but including the pressure and the remaining $\nabla \mu$ terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the 2D Log-Elastographic central difference scheme.
Figure 5.13: For example IV(8) where the contrast between the background and the maximum value in the inclusion is 2 to 16: (a) is the exact scaled shear modulus; (b) is the recovered scaled shear modulus obtained from the Direct Inversion method (I) with a locally constant assumption; (c) is the recovered scaled shear modulus obtained by neglecting the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ terms but including the remaining $\nabla \tilde{\mu}$ terms; (d) is the recovered shear modulus obtained by neglecting the $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ term but including the pressure and the remaining $\nabla \tilde{\mu}$ terms; (e) is the recovered scaled shear modulus obtained from the standard upwind scheme; (f) is the recovered scaled shear modulus obtained from the central difference scheme; (g) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic upwind scheme and (h) is the recovered scaled shear modulus obtained from the $2D$ Log-Elastographic central difference scheme.
2D Log-Elastographic nonlinear algorithms recover the scaled shear modulus with more accurate shape and size of both the single inclusion and multiple inclusions. This improvement occurs when we have either low or high contrast between the background medium and the maximum value in the inclusion. Compared with the standard upwind scheme and central difference scheme, the 2D Log-Elastographic algorithms improve the quality of the reconstructions in the background and in the regions where the contrast of $\tilde{\mu}$ is high. Note that the recovered inclusion amplitudes obtained from the 2D Log-Elastographic central difference algorithm exhibit slightly more undershooting than the recoveries obtained from the 2D Log-Elastographic upwind scheme.

Besides the scaled shear modulus, we also recover the pressure in the 2D Log-Elastographic methods. In Figure 5.14 and Figure 5.15 we compare the pressure, obtained from the forward calculation for the above examples 4 and 5, to the pressure we recover in the 2D Log-Elastographic upwind method for the same examples. These images show that besides the scaled shear modulus we can also recover the pressure at the same time, and the recovered pressure is similar to the pressure obtained from the forward simulation.

### 5.2.3 Recoveries with Inversion Algorithms Using Noisy Data

To test the stability and robustness of our 2D Log-Elastographic algorithms, we add 10% and 20% Gaussian random noise to the displacement data respectively

$$u_{noisy}(x, y, t) = u(x, y, t) + \gamma u_{max} \text{rand}(x, y, t)$$

where $u_{max}$ is the maximum displacement amplitude throughout the computational domain, $\gamma$ is the noise level, and $\text{rand}(x, y, t)$ is a random vector generated in Matlab from the normal distribution with mean zero and variance one. We apply the same set of inverse algorithms to this data set with 10% and 20% noise ($\gamma = 0.1$ and 0.2) for all examples (I)-(IV) in Section 5.2.2.

Again we need to differentiate the data. But if there is noise in the data, the error can be strongly amplified in the standard differentiation process, which means that the differentiation process is ill-posed. Since we now have noise in the
Figure 5.14: For example II(4), (a) is the real part of the pressure simulated with forward problem algorithm after taking the Fourier transform; (b) is the real part of the recovered pressure $\tilde{p}$ from the 2D Log-Elastographic upwind scheme; (c) is the imaginary part of the pressure simulated with forward problem algorithm after taking the Fourier transform; (d) is the imaginary part of the recovered pressure $\tilde{p}$ from the 2D Log-Elastographic upwind scheme.

data, we apply an averaging method introduced by Anderssen and Hegland, see [1], to compute the derivatives of the noisy displacement. This method averages the displacement over local windows and following that averaging procedure is then essentially based on a central difference scheme (or mid-point rule) to approximate the derivatives. The step size for the central difference is related to the local window sizes and acts as a regularization parameter to control the tradeoff between the accuracy and smoothness of the derivatives.

The recovered scaled shear moduli are shown in Figures 5.16-5.23. In each figure, (a),(e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ terms are neglected but the remaining $\nabla \tilde{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are
the recoveries obtained from the 2D Log-Elastographic upwind scheme. Comparing those images, you can see that most of time, the inclusions are either missing or deformed in the recoveries obtained either from the Direct Inversion (I) or from solving the acoustic equation with the Acoustic Log-Elastographic algorithm; while all the inclusions in the recoveries obtained from the 2D Log-Elastographic algorithm still can be recognized with better shapes and contrasts. Furthermore, the recoveries obtained from the 2D Log-Elastographic algorithm have improved backgrounds, especially when the contrasts of the scaled shear modulus is high, compared with the recoveries obtained from the standard upwind scheme. This also shows that the 2D Log-Elastographic method is robust with respect to noise.
Figure 5.16: For example I(1), the contrast here between the background and the maximum value in the inclusion is 2 to 6 and the noise level is 10% for (a)-(d), 20% for (e)-(h); (a), (e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ terms are neglected but the remaining $\nabla \tilde{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.17: For example I(2), the contrast here between the background and the maximum value in the inclusion is 2 to 18 and the noise level is 10% for (a)-(d), 20% for (e)-(h); (a), (e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\tilde{\mu} \nabla \tilde{u})$ terms are neglected but the remaining $\nabla \tilde{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.18: For example II(3), the contrast here between the background and the maximum value in the inclusion is 2 to 4 and the noise level is 10% for (a)-(d), 20% for (e)-(h); (a), (e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u})$ terms are neglected but the remaining $\nabla \tilde{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.19: For example II(4), the contrast here between the background and the maximum value in the inclusion is 2 to 12 and the noise level is 10% for (a)-(d), 20% for (e)-(h): (a), (e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\hat{\mu} \nabla \hat{u}^T)$ terms are neglected but the remaining $\nabla \hat{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.20: For example III(5), the contrast here between the background and the maximum value in the inclusion is 2 to 4 and the noise level is 10\% for (a)-(d), 20\% for (e)-(h); (a),(e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u})$ terms are neglected but the remaining $\nabla \tilde{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.21: For example III(6), the contrast here between the background and the maximum value in the inclusion is 2 to 12 and the noise level is 10% for (a)-(d), 20% for (e)-(h); (a), (e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and \( \nabla \cdot (\tilde{\mu} \nabla \hat{u}) \) terms are neglected but the remaining \( \nabla \tilde{\mu} \) terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.22: For example IV(7), the contrast here between the background and the maximum value in the inclusion is 2 to 4 and the noise level is 10% for (a)-(d), 20% for (e)-(h); (a), (e) are the recoveries obtained from the Direct Inversion method (I); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and \( \nabla \cdot (\tilde{\mu} \nabla \hat{u}) \) terms are neglected but the remaining \( \nabla \tilde{\mu} \) terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Figure 5.23: For example IV(8), the contrast here between the background and the maximum value in the inclusion is 2 to 16 and the noise level is 10% for (a)-(d), 20% for (e)-(h); (a),(e) are the recoveries obtained from the Direct Inversion method (l); (b), (f) are the recoveries obtained from solving the acoustic equation where the pressure and $\nabla \cdot (\tilde{\mu} \nabla \hat{u}^T)$ terms are neglected but the remaining $\nabla \tilde{\mu}$ terms are included; (c), (g) are the recoveries obtained from the standard upwind scheme and (d), (h) are the recoveries obtained from the 2D Log-Elastographic upwind scheme.
Table 5.1: Convergence Results for the 2D Log-Elastographic Upwind Scheme for Example I(1); the contrast between the background and the maximum value inside the inclusion is 2 to 6.

<table>
<thead>
<tr>
<th>$\Delta x (= \Delta y)$</th>
<th>h</th>
<th>$h/2$</th>
<th>$h/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_2$ Error</td>
<td>12.402e-2</td>
<td>6.1507e-2</td>
<td>2.2596e-2</td>
</tr>
<tr>
<td>$L_{\infty}$ Error</td>
<td>15.4165e-2</td>
<td>7.4786e-2</td>
<td>2.8738e-2</td>
</tr>
</tbody>
</table>

5.2.4 Numerical Tests of Convergence Rate of the 2D Log-Elastographic Algorithms

We establish the first order convergence rate for the 2D Log-Elastographic upwind scheme and the almost first order convergence rate for the 2D Log-Elastographic central difference scheme in Chapter 4. In this section, we test their convergence rates numerically with the examples given above, example 1,2. In the first example, the contrast between the scaled shear modulus in the background and the maximum value inside the inclusion is low. While in the second example, the contrast is higher. We first do the forward simulation to generate the displacement data on a 100 by 100 coarse grid and use this data to calculate the coefficients in the inverse problem model. After reconstructing the scaled shear modulus with the displacement data on the coarse grid, we then interpolate the coefficients on a two times finer grid and recover the scaled shear modulus on this finer grid. We repeat this procedure three times and the recoveries are then compared with the solution on the finest grid. From the results showed in Tables 5.1-5.4, we can clearly see that the recoveries obtained from the 2D Log-Elastographic upwind algorithm converge at the rate of $O(h)$ and those from the 2D Log-Elastographic central difference scheme converge almost at the rate of $O(h)$, which is consistent with the theoretical results we obtained in Chapter 4.
Table 5.2: Convergence Results for the 2D Log-Elastographic Central Difference Scheme for Example I(1); the contrast between the background and the maximum value inside the inclusion is 2 to 6.

<table>
<thead>
<tr>
<th>Δx (= Δy)</th>
<th>h</th>
<th>h/2</th>
<th>h/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>L₂ Error</td>
<td>15.6678e-2</td>
<td>7.4894e-2</td>
<td>3.3330e-2</td>
</tr>
<tr>
<td>L∞ Error</td>
<td>21.2124e-2</td>
<td>7.4786e-2</td>
<td>4.6076e-2</td>
</tr>
</tbody>
</table>

Table 5.3: Convergence Results for the 2D Log-Elastographic Upwind Scheme for Example I(2); the contrast between the background and the maximum value inside the inclusion is 2 to 18.

<table>
<thead>
<tr>
<th>Δx (= Δy)</th>
<th>h</th>
<th>h/2</th>
<th>h/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>L₂ Error</td>
<td>27.7273e-2</td>
<td>14.0965e-2</td>
<td>5.3793e-2</td>
</tr>
</tbody>
</table>

Table 5.4: Convergence Results for the 2D Log-Elastographic Central Difference Scheme for Example I(2); the contrast between the background and the maximum value inside the inclusion is 2 to 18.

<table>
<thead>
<tr>
<th>Δx (= Δy)</th>
<th>h</th>
<th>h/2</th>
<th>h/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>L∞ Error</td>
<td>71.0722e-2</td>
<td>34.1008e-2</td>
<td>12.0447e-2</td>
</tr>
</tbody>
</table>
CHAPTER 6
Numerical Examples on Experimental Data

In this chapter, we apply the 2D Log-Elastographic upwind algorithm to recover the scaled shear modulus with the MR measured displacement data in the liver and the spleen. This data set is provided by R. L. Ehman’s lab from Mayo Clinic. In their experiment, low-amplitude mechanical waves at 60 Hz were generated in the abdomen during imaging by using an acoustic driver device placed on the anterior body wall. The waves propagate along the diagonal direction of the field of view (FOV). 3D displacement data was then obtained by using a 2-dimensional gradient echo MR elastography sequence to collect axial wave images in three different orthogonal directions, see [54] and [69]. The FOV is a single 2D plane, whose size is $38 \times 38$ cm with $96 \times 96$ pixels. This displacement is given at 4 equally spaced times in a single period. The amplitude of the wave is measured in terms of microns.

The 3D displacement vector contains three components: two in-plane components $u_1$, $u_2$ and one out-of-plane component $u_3$. We first apply a coordinate transformation to obtain two new components from $u_1$ and $u_2$. One of these new components is along the wave propagation direction, say $u_p$, and the other one is orthogonal to the wave propagation direction, say $u_m$. They both are still in the plane and hence satisfy the 2D plane strain elastic system. Moreover, for this data set the amplitudes of all the components are approximately the same. Therefore, in our inversion method, we first assume one of the two components, $u_p$, is the leading component and then the other, $u_m$. In our Log-Elastographic upwind algorithm, we treat the variable in the wave propagation direction as a time like variable. But here the wave propagates along the diagonal. Therefore, we do the inversion in two different ways: first we treat the $x$ variable as a time like variable and then we treat the $y$ variable as a time like variable. Altogether there are 4 different possible combinations to do the full inversion with the 2D Log-Elastographic algorithm. Afterwards, we take the geometric average of all these recoveries.

In the MR measured displacement data of the liver and spleen and the regions
bounding them, there exist very noisy regions, particularly in the area surrounding those two organs. Therefore we separate the regions where we will do recovery, eliminating the surrounding regions and separately recovering the scaled shear modulus \( \tilde{\mu} \) in the liver and spleen. Figure 6.1 shows the MR images of these two organs where we do our recovery for the patient and the healthy volunteer.

![MR images](image)

**(Figure 6.1)**: (a) is the MR image of the tissue in the image plane cross section for the patient; (b) is the MR image in the liver for the patient; (c) is the MR image in the spleen for the patient; (d) is the MR image of the tissue in the image plane cross section for the healthy volunteer; (e) is the MR image in the liver for the healthy volunteer; (f) is the MR image in the spleen for the healthy volunteer.

As for the boundary, we still require Dirichlet boundary condition on the boundaries of the computational domain when we have the inflow condition, where we choose the boundary values to be the results obtained from the Direct Inversion method (I). Here the boundaries of the above two computational domains are curved and we compute only inside the boundary curves. Since the 2D Log-Elastographic upwind method is a marching method, when we add grid points due to the curved boundary we set the values of \( \tilde{\mu} \) at those points to be the values obtained from the Direct Inversion method (I).

The material here, i.e., the liver and spleen tissue, is viscoelastic, so the shear modulus in the liver and spleen is complex. This is different from our previous
examples on synthetic data, where the shear moduli are chosen to be real. Here we need to consider the complex Fourier transformed displacement because we are recovering the complex shear modulus. Therefore, before applying the 2D Log-Elastographic algorithm, we first let \( \hat{u} = M e^{k\phi} \), where \( k = \sqrt{-1} \), \( M = |\hat{u}| \) is the magnitude and \( \phi = \angle(\hat{u}) \) is the phase. Note that \( u \) here can denote any of the above two in-plane components: \( u_p \) and \( u_m \). \( \hat{u} \) is its Fourier transform. For different components, the corresponding magnitudes and phases are different in general. Substitution into the 2D elastic system (3.3) yields

\[
\mathbf{v}_x + F_{mp}^1 \mathbf{v}_y + F_{mp}^2 \mathbf{v} + F_{mp}^3 = \mathbf{0}
\]

where, using somewhat mixed notation,

\[
F_{mp}^1 = \begin{pmatrix}
\frac{2\bar{u}_2}{\bar{u}_2 + \bar{u}_1} \exp(k(\phi_1 - \phi_2)) & \frac{\exp(-k\phi_2)}{\bar{u}_2 + \bar{u}_1} \\
\bar{u}_2 \exp(k\phi_2) + \bar{u}_1 & -\frac{4\bar{u}_1 \bar{u}_2 \exp(k\phi_1)}{\bar{u}_2 + \bar{u}_1} \exp(k(\phi_1 - \phi_2))
\end{pmatrix},
\]

\[
F_{mp}^2 = \begin{pmatrix}
\frac{\Delta \bar{u}_2}{\bar{u}_2 + \bar{u}_1} \exp(k(\phi_1 - \phi_2)) & 0 \\
\Delta \bar{u}_1 \exp(k\phi_1) - \frac{\omega^2 M_1}{2\bar{u}_1} \bar{u}_2 & 0
\end{pmatrix},
\]

\[
F_{mp}^3 = \begin{pmatrix}
\frac{\omega^2 M_2}{\bar{u}_2 + \bar{u}_1} \exp(k(\phi_1 - \phi_2)) & \omega^2 M_1 \exp(k\phi_1) - \frac{\omega^2 M_2}{\bar{u}_2 + \bar{u}_1} \bar{u}_2 \\
\omega^2 M_1 \exp(k\phi_1) - \frac{\omega^2 M_2}{\bar{u}_2 + \bar{u}_1} \bar{u}_2 & \omega^2 M_2 \exp(k\phi_1)
\end{pmatrix},
\]

with

\[
\bar{u}_{1,x} = M_{1,x} + kM_1 \phi_{1,x}, \quad \bar{u}_{1,y} = M_{1,y} + kM_1 \phi_{1,y}, \\
\bar{u}_{2,x} = M_{2,x} + kM_2 \phi_{2,x}, \quad \bar{u}_{2,y} = M_{2,y} + kM_2 \phi_{2,y}, \\
\Delta \bar{u}_1 = \Delta M_1 - M_1 |\nabla \phi_1|^2 + k(2\nabla M_1 \cdot \nabla \phi_1 + M_1 \Delta \phi_1), \\
\Delta \bar{u}_2 = \Delta M_2 - M_2 |\nabla \phi_2|^2 + k(2\nabla M_2 \cdot \nabla \phi_2 + M_2 \Delta \phi_2).
\]

We solve this system with the 2D Log-Elastographic upwind scheme for the scaled shear modulus and the pressure. In order to solve this system, we need to calculate the derivatives of the magnitude \( M \) and the phase \( \phi \). Relatively speaking, from
our observations, the magnitudes and the phases of those two components \( u_p \) and 
\( u_m \) throughout our selected computational regions change smoothly, except at some 
places where there are jumps in the phases. Therefore, we take the following steps 
to calculate the derivatives of the magnitude and the phase. First, for the phase, 
we shift the interval for the definition of the phase to eliminate jumps; second, we 
do local median filtering of the magnitude and the phase; third, we calculate the 
derivatives at each interior point by taking the median of the second order central 
difference approximation and two second order one-sided approximations; then on 
the boundary we use only a one-sided second order approximation to the derivative 
at each point.

For the experimental data, the expected shear modulus is complex valued. 
In this case we would have arbitrarily high exponential error growth. To control 
this unwanted growth, we apply an additional filtering step after each sweeping 
step. Basically, we filter out the high frequency content of the computed solution 
at the end of each sweeping step. Our preliminary realization is as follows: First at 
each sweeping step we take the Fast Fourier Transform with respect to the transverse 
space variable, in the direction orthogonal to the sweeping direction, of the computed 
solution \( \{\mu_{i,j}, p_{i,j}\}_{j=1}^N \); we obtain the vector \( \{V_{i,j}\}_{j=1}^N = FFT(\{\mu_{i,j}, p_{i,j}\}_{j=1}^N) \). Then 
we filter it with the formula given below

\[
(fV_i)_j = \begin{cases} 
  V_{i,j}, & \text{if } |j| \leq L \\
  0, & \text{otherwise}
\end{cases}
\]

where \( L \) is half of the total length, and \( V_i = \{V_{i,j}\}_{j=1}^N \) is the computed solution after 
FFT at present step. Lastly we take the inverse Fast Fourier Transform of \( fV_i \) to 
obtain the final solution at the present step. Repeat this procedure at the next step, 
etc.. Generally, the frequency content \( \omega \) and the total length \( l \) of the vector under 
FFT have the relationship given by \( \omega(k) = 2\pi * (k - 1)/l \), where \( k \) is the index, 
therefore by keeping half of the total length of the vector under FFT is equivalent 
to keeping up to the same frequency content at each step.

Figure 6.2 shows the recoveries we obtained for the liver and spleen of the 
patient using (6.1) with our 2D Log-Elastographic upwind algorithm together with
the filtering procedure. Figure 6.3 shows the recoveries we obtained for the liver and spleen of the healthy volunteer. In each of these figures, (a), (b), (e) and (f) are the recovered shear moduli obtained from sweeping along the $x$ direction; (c), (d), (g) and (h) are the recovered shear moduli obtained from sweeping along the $y$ direction, where (a), (c), (e) and (g) are the recoveries from $u_p$, $u_m$ where $u_p$ is the leading component; (b), (d), (f) and (h) are the recoveries from $u_p$, $u_m$ where $u_m$ is the leading component. In Figure 6.4, we show the geometric average of the recoveries obtained from sweeping along either $x$ direction or along $y$ direction for the liver and spleen of the patient and the healthy volunteer. From these images, you can see that the recovery obtained from taking the geometric average of all those recoveries obtained from sweeping along the $x$ direction, and the recovery obtained from the geometric average of the recoveries obtained by sweeping along the $y$ direction are consistent. Furthermore, the recoveries can be divided into two groups. In either of these groups, the one obtained from sweeping along the $x$ direction with a component as the leading (non-leading) component is very similar to the one obtained from sweeping along the $y$ direction with the same component as the non-leading (leading) component.

In addition to setting the values of $\tilde{\mu}$ at the points due to the curved boundary as the results from the Direct Inversion method (I), we also set the values of $\tilde{\mu}$ at those points as the results from the Direct Inversion (II). The images of the recovered shear wave speed squared, see Figure 6.5, 6.6, show that we obtain very similar results.

We also obtain recoveries with the Direct Inversion method (I) and (II). In the Direct Inversion method (I) we compute the geometric average of the three direct inversion recoveries obtained from using each of the three components respectively. In the Direct Inversion method (II) we take the curl of the two in-plane components. Figure 6.5 shows the recovered wave speed squared in the liver and the spleen for the patient from the 2D Log-Elastographic method (after taking geometric average of all recoveries) with the Direct Inversion method (I) as the boundary conditions and with the Direct Inversion method (II) as the boundary conditions, separately, the Direct Inversion method (I) and the Direct Inversion method (II). Figure 6.6
Figure 6.2: (a)-(h) are recovered scaled shear wave speed squared in the liver for the patient obtained from sweeping along the x direction: (a) is the recovery from $u_p$, $u_m$ where $u_p$ is the leading component; (b) is the recovery from $u_p$, $u_m$ where $u_m$ is the leading component; (c)-(d) are recovered scaled shear wave speed squared in the liver for the patient obtained from sweeping along the y direction: (c) is the recovery from $u_p$, $u_m$ where $u_p$ is the leading component; (d) is the recovery from $u_p$, $u_m$ where $u_m$ is the leading component; (e)-(f) are recovered scaled shear wave speed squared in the spleen for the patient obtained from sweeping along the x direction: (e) is the recovery from $u_p$, $u_m$ where $u_p$ is the leading component; (f) is the recovery from $u_p$, $u_m$ where $u_m$ is the leading component; (g)-(h) are recovered scaled shear wave speed squared in the spleen for the patient obtained from sweeping along the y direction: (g) is the recovery from $u_p$, $u_m$ where $u_p$ is the leading component; (h) is the recovery from $u_p$, $u_m$ where $u_m$ is the leading component.
Figure 6.3: (a)-(b) are recovered scaled shear wave speed squared in the liver for the healthy volunteer obtained from sweeping along the $x$ direction: (a) is the recovery from $u_{p}, u_{m}$ where $u_{p}$ is the leading component; (b) is the recovery from $u_{p}, u_{m}$ where $u_{m}$ is the leading component; (c)-(d) are recovered scaled shear wave speed squared in the liver for the healthy volunteer obtained from sweeping along the $y$ direction: (c) is the recovery from $u_{p}, u_{m}$ where $u_{p}$ is the leading component; (d) is the recovery from $u_{p}, u_{m}$ where $u_{m}$ is the leading component; (e)-(f) are recovered scaled shear wave speed squared in the spleen for the healthy volunteer obtained from sweeping along the $x$ direction: (e) is the recovery from $u_{p}, u_{m}$ where $u_{p}$ is the leading component; (f) is the recovery from $u_{p}, u_{m}$ where $u_{m}$ is the leading component; (g)-(h) are recovered scaled shear wave speed squared in the spleen for the healthy volunteer obtained from sweeping along the $y$ direction: (g) is the recovery from $u_{p}, u_{m}$ where $u_{p}$ is the leading component; (h) is the recovery from $u_{p}, u_{m}$ where $u_{m}$ is the leading component.
Figure 6.4: (a) is the average of the recoveries from sweeping along the \( x \) direction in the liver of the patient; (b) is the average of the recoveries from sweeping along the \( y \) direction in the liver of the patient; (c) is the average of the recoveries from sweeping along the \( x \) direction in the spleen of the patient; (d) is the average of the recoveries from sweeping along the \( y \) direction in the spleen of the patient; (e) is the average of the recoveries from sweeping along the \( x \) direction in the liver of the healthy volunteer; (f) is the average of the recoveries from sweeping along the \( y \) direction in the liver of the healthy volunteer; (g) is the average of the recoveries from sweeping along the \( x \) direction in the spleen of the healthy volunteer; (h) is the average of the recoveries from sweeping along the \( y \) direction in the spleen of the healthy volunteer.
shows the recovered wave speed squared in the liver and the spleen for a healthy volunteer from the 2D Log-Elastographic method (after taking geometric average of all recoveries) with the Direct Inversion method (I) as the boundary conditions and with the Direct Inversion method (II) as the boundary conditions, separately, the Direct Inversion method (I) and the Direct Inversion method (II).

To compare the results from the Direct Inversion method (I) and the 2D Log-Elastographic algorithm, we have also calculated: (1) Average Difference: The average value of the difference between the Direct Inversion (I) result and the 2D Log-Elastographic result; (2) $L^1$ Difference: The $L^1$ norm (sum of the absolute value of the differences, pixel by pixel divided by the number of pixels) of the difference between the Direct Inversion (I) result and the 2D Log-Elastographic result; (3) $L^2$ Difference: The root mean square difference between the Direct Inversion (I) result and the 2D Log-Elastographic result. The results are tabulated for the in vivo data in Table 6.1 when no cutoff is applied to the imaging functionals. In Figure 6.5 and Figure 6.6, we have put all the images on the same scale, i.e., we cut off the values larger than 7. In this case, we also calculated the three differences mentioned above. Table 6.2 shows the results, where the differences between the recoveries from the Direct Inversion method (I) and the recoveries from the 2D Log-Elastographic algorithm become smaller than the differences when there is no cutoff. This shows that most of the differences occur in the places where the recovered shear wave speed squared is larger than 7. At the places where the recovered shear wave speed squared is less or equal to 7, the results from the Direct Inversion Method (I) undershoot the results from the 2D Log-Elastographic Method in the liver and spleen of the patient, while overshoot the results from the 2D Log-Elastographic Method in the

<table>
<thead>
<tr>
<th>Cases</th>
<th>Average Difference</th>
<th>$L^1$ Difference</th>
<th>$L^2$ Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>liver of the patient</td>
<td>0.0066</td>
<td>1.3556</td>
<td>4.2302</td>
</tr>
<tr>
<td>spleen of the patient</td>
<td>-0.1970</td>
<td>1.2790</td>
<td>1.6796</td>
</tr>
<tr>
<td>liver of the healthy volunteer</td>
<td>0.1522</td>
<td>0.8766</td>
<td>1.6815</td>
</tr>
<tr>
<td>spleen of the healthy volunteer</td>
<td>0.5006</td>
<td>1.0383</td>
<td>4.7893</td>
</tr>
</tbody>
</table>
Figure 6.5: (a) The recovered speed squared in the liver for the patient with the 2D Log-Elastographic method (Direct Inversion (I) results as B.C.s); (b) The recovered speed squared in the spleen for the patient with the 2D Log-Elastographic method (Direct Inversion (I) results as B.C.s); (c) The recovered speed squared in the liver for the patient with the 2D Log-Elastographic method (Direct Inversion (II) results as B.C.s); (d) The recovered speed squared in the spleen for the patient with the 2D Log-Elastographic method (Direct Inversion (II) results as B.C.s); (e) The recovered speed squared in the liver for the patient with the Direct Inversion method (I); (f) The recovered speed squared in the spleen for the patient with the Direct Inversion method (I); (g) The recovered speed squared in the liver for the patient with the Direct Inversion method (II) by taking curl; (h) The recovered speed squared in the spleen for the patient with the Direct Inversion method (II) by taking curl.
Figure 6.6: (a) The recovered speed squared in the liver for the healthy volunteer with the 2D Log-Elastographic method (Direct Inversion (I) results as B.C.s); (b) The recovered speed squared in the spleen for the healthy volunteer with the 2D Log-Elastographic method (Direct Inversion (I) results as B.C.s); (c) The recovered speed squared in the liver for the healthy volunteer with the 2D Log-Elastographic method (Direct Inversion (II) results as B.C.s); (d) The recovered speed squared in the spleen for the healthy volunteer with the 2D Log-Elastographic method (Direct Inversion (II) results as B.C.s); (e) The recovered speed squared in the liver for the healthy volunteer with the Direct Inversion method (I); (f) The recovered speed squared in the spleen for the healthy volunteer with the Direct Inversion method (I); (g) The recovered speed squared in the liver for the healthy volunteer with the Direct Inversion method (II) by taking curl; (h) The recovered speed squared in the spleen for the healthy volunteer with the Direct Inversion method (II) by taking curl.
liver and spleen of the healthy volunteer.

**Table 6.2: Average difference, $L^1$ and $L^2$ differences between the Direct Inversion (I) image and the 2D Log-Elastographic image when cutoff is applied to the imaging functional.**

<table>
<thead>
<tr>
<th>Cases</th>
<th>Average Difference</th>
<th>$L^1$ Difference</th>
<th>$L^2$ Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>liver of the patient</td>
<td>-0.2435</td>
<td>1.0607</td>
<td>1.3880</td>
</tr>
<tr>
<td>spleen of the patient</td>
<td>-0.2082</td>
<td>1.1675</td>
<td>1.5105</td>
</tr>
<tr>
<td>liver of the healthy volunteer</td>
<td>0.1108</td>
<td>0.8352</td>
<td>1.1161</td>
</tr>
<tr>
<td>spleen of the healthy volunteer</td>
<td>0.2359</td>
<td>0.7736</td>
<td>1.0774</td>
</tr>
</tbody>
</table>

The scaled shear wave speed we obtained with the data from the healthy volunteer is comparable with the recoveries in [26], where their results are obtained with the liver of five healthy volunteers. In [54] and [69], the authors have recovered the shear modulus for the liver and spleen of patients and healthy volunteers. The mean values of our recoveries with the 2D Log-Elastographic algorithm are $3.7171 \text{ m}^2/\text{s}^2$ for the liver of the patient, $3.4319 \text{ m}^2/\text{s}^2$ for the spleen of the patient, $2.6390 \text{ m}^2/\text{s}^2$ for the liver of the healthy volunteer and $1.7915 \text{ m}^2/\text{s}^2$ for the spleen of the healthy volunteer. These values times the density $\approx 10^3 \text{ Kg/m}^3$ are in the range, $1.77-2.85\text{kPa}$, of those in [69] for the healthy volunteers, and in the range, $2.76-12.01\text{kPa}$, for the group of patients, and also are comparable with those for the patients with fibrosis stage 2, where the range is $3.2\pm0.8\text{kPa}$.

Besides the scaled shear wave speed squared, we have also calculated the attenuation with formula in [17]. In Figure 6.7, we show the attenuation in the liver obtained with the 2D Log-Elastographic algorithm, where we use the results from the Direct Inversion method (I) as the boundary conditions, for the patient and the volunteer. But we do not include here additional comparisons for this biomechanical property.
Figure 6.7: (a) The recovered attenuation in the liver for the patient; (b) The recovered attenuation in the spleen for the patient; (c) The recovered attenuation in the liver for the healthy volunteer; (d) The recovered attenuation in the spleen for the healthy volunteer. All the images are obtained from the 2D Log-Elastographic method with the Direct Inversion method (I) results as B.C.s.
CHAPTER 7
Conclusions and Future Work

Using the 2D plane strain elastic model we have developed a 2D Log-Elastographic nonlinear/linear algorithm to solve the inverse problem, i.e., to find the scaled shear modulus \( \tilde{\mu} \) and the scaled pressure \( \hat{\hat{p}} \) simultaneously with the given 2D displacement data \( \hat{u} \) at single frequency. We treat the 2D plane strain elastic system with \( \hat{u} \) given as a first order p.d.e. system of evolution type for \( \tilde{\mu} \) and \( \hat{\hat{p}} \), and create a finite difference based marching algorithm.

This first order p.d.e. system has a zeroth order derivative term in \( \tilde{\mu} \), so when coefficient matrices are real and have real eigenvalues, the solution to this first order p.d.e. system may have exponential growth if the eigenvalues of the coefficient matrix in front of the zeroth order derivative term of the shear modulus have the wrong sign; and when coefficient matrices are complex and have complex eigenvalues, the solution will have exponential growth of arbitrarily high order. In the case of real eigenvalues, in order to control the potential exponential growth in the error without requiring a fine discretization, we use a Log-Elastographic representation for the shear modulus to transform the linear system for the shear modulus and the pressure into a nonlinear system for the logarithm of the shear modulus and the pressure. Then upwind or Lax-Friedrichs linear finite difference discretizations are applied. After that, the Log-Elastographic algorithm is established by taking the exponential of the resulting discretized system and selectively linearizing some, but not all, of the resulting exponential terms.

In this thesis, we apply the standard upwind scheme and the stable central difference scheme (Lax-Friedrichs) to approximate the transformed system. When using the standard upwind scheme, the eigenvalues and eigenvector matrix of the coefficient matrix in front of the \( y \)-derivative term here in the transformed system need to be computed at each point in the computational domain. Since the coefficient matrix in front of the \( y \)-derivative term contains the unknown, \( \tilde{\mu} \), in our case, this calculation can bring additional inaccuracies. However we established a relationship...
between the eigenvalues, eigenvectors and components of the coefficient matrix in
the original system and the coefficient matrix in the transformed system. This re-
relationship enables us to effectively precompute the eigenvalues and eigenvectors of
the coefficient matrices and so the instabilities can be avoided.

The full 2D plane strain elastic system contains a term which is nonlinear in
the unknowns, \( \tilde{\mu}, \hat{p} \) and contains the multiplicative factor \( \rho/\lambda \). We show that this
term can be neglected. To establish this we compute our solutions, \( \tilde{\mu}, \hat{p} \), with this
nonlinear term and without this nonlinear term and exhibit the difference in the two
computed solutions. The error is of order \( 10^{-6} \) which is the same order as \( \rho/\lambda \). To
compute the solution when the nonlinear term is included, we apply an alternating
idea which is not needed when the nonlinear term is neglected.

We establish first order convergence results for the 2D Log-Elastographic al-
gorithm when the eigenvalues of coefficient matrices are real. We show numerical
examples to exhibit these same properties.

Additional numerical results have also been presented. First we test the Log-
Elastographic algorithm with synthetic displacement data obtained by solving the
forward 2D plane strain elastic system. When solving the forward problem we
applied a Perfectly Matched Layer method to prevent artificial reflections when
waves propagate from the interior domain into the boundary. The single frequency
excitation is a line source on a line interior to the domain. Our examples have
multiple inclusions of different sizes and shapes and with different contrasts with
the background.

The 2D Log-Elastographic algorithm recovers the shear wave speed together
with the pressure in the presence of single inclusion or multiple inclusions with ei-
ther high or low contrast between the background and the maximum value inside the
inclusions. Numerical examples show that: 1) pressure cannot be neglected and by
recovering pressure simultaneously with the shear modulus prevents shear modulus
reconstruction undershooting that occurs when the pressure terms are neglected; 2)
the 2D Log-Elastographic algorithm improves the quality of the recovered images
including size, shape and contrast of the inclusions, compared with the reconstruc-
tions obtained from applying the direct inversion method when all the derivative
terms of \( \tilde{\mu} \) and the pressure term are neglected, and with the reconstructions obtained from applying the acoustic Log-Elastographic method when the pressure and \( \nabla \cdot (\tilde{\mu} \nabla \hat{u}^T) \) terms are neglected; 3) compared with the standard upwind scheme and central difference scheme, the 2D Log-Elastographic algorithm improves the quality of the reconstructions in the background and in the regions where the contrast of \( \tilde{\mu} \) is high; (4) using Direct Inversion (I) results as an initial condition in the 2D Log-Elastographic algorithm yields reliable reconstructions; and (5) the results of the 2D Log-Elastographic algorithm show that when the shear modulus is real the possible exponential growth in the error does not occur.

Initial results with MRE generated displacement data, obtained from Richard Ehman’s laboratory at Mayo Clinic, in the liver and spleen have also been obtained. In this case, the scaled shear modulus and the coefficients are both complex and an additional filtering step is made because arbitrarily high exponential growth in the numerical error can occur. We compute our images both when the boundary data is selected from Direct Inversion Method (I) and when the boundary data is selected from Direct Inversion Method (II).

In the future research, we will target establishing the stability when the coefficient matrices and the coefficient matrix eigenvalues are complex where in this case we add an additional filtering step. We will also add the viscoelastic effect of biological tissues in our mathematical model to do the forward simulation and inversion.
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APPENDIX A
Proofs of Theorems 4.2, 4.4 and 4.6

In this appendix, we give the proofs of Theorems 4.2, 4.4, and 4.6 stated in Chapter 4 for the central difference scheme for first order linear p.d.e. solution calculations and for the 2D Log-Elastographic central difference schemes.

A.1 Proof of Theorem 4.2

Proof. To apply the Von Neumann method of Fourier stability analysis, let

\[ \hat{V}_{i+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-kj\xi} V_{i+1,j} \]

where \( k = \sqrt{-1} \). Assume that the coefficient matrices are constant, then we can write

\[ \hat{V}_{i+1}(\xi) = (I + \Delta x E^2)^{-1} \left( \bar{G}(\xi) \hat{V}_i(\xi) - \Delta x E^3 e^{-kj\xi} \right) \]

where \( \bar{G}(\xi) \) is a 2 \( \times \) 2 matrix given by

\[ \bar{G}(\xi) = \frac{1}{2} \left( I - \frac{\Delta x}{\Delta y} E^1 \right) e^{k\xi} + \frac{1}{2} \left( I + \frac{\Delta x}{\Delta y} E^1 \right) e^{-k\xi} \].

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So after \( n + 1 \) iterations,

\[
\hat{V}_{n+1} = (I + \Delta x E^2)^{-1} \tilde{G}(\xi) \hat{V}_n - \Delta x (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi} \\
= (I + \Delta x E^2)^{-1} \tilde{G}(\xi) ((I + \Delta x E^2)^{-1} \tilde{G}(\xi) \hat{V}_{n-1} - \Delta x (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi}) \\
- \Delta x (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi} \\
= ((I + \Delta x E^2)^{-1} \tilde{G}(\xi))^2 \hat{V}_{n-1} \\
- \Delta x ((I + \Delta x E^2)^{-1} \tilde{G}(\xi) + I) (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi} \\
= \ldots \\
= ((I + \Delta x E^2)^{-1} \tilde{G}(\xi))^{n+1} \hat{V}_0 \\
- \Delta x \left( ((I + \Delta x E^2)^{-1} \tilde{G}(\xi))^n + \cdots + I \right) (I + \Delta x E^2)^{-1} E^3 e^{-kj\xi},
\]

implying

\[
\|\hat{V}_{n+1}\|_2 \leq \|((I + \Delta x E^2)^{-1} \tilde{G})^{n+1}\|_2 \|\hat{V}_0\|_2 + \Delta x \left( \|((I + \Delta x E^2)^{-1} \tilde{G})^n\|_2 + \|((I + \Delta x E^2)^{-1} \tilde{G})^{n-1}\|_2 + \cdots + 1 \right) \|I + \Delta x E^2\|_2 \|E^3\|_2.
\]

In order to establish our stability result, we need to find non-negative constants \( K \) and \( \alpha \) such that

\[
\|((I + \Delta x E^2)^{-1} \tilde{G})^l\|_2 \leq Ke^{\alpha l \Delta x}
\]

for any \( l \).

From Lemma 4.1, \( \|\tilde{G}^n\|_2 \leq C_1 C_2 \), where \( C_1, C_2 \) are constants satisfying \( \|S\|_2 \leq C_1, \|S^{-1}\|_2 \leq C_2 \). On the other hand, from the proof of Theorem 4.1 we know that

\[
(I + \Delta x E^2)^{-1} = \frac{1}{1 + a \Delta x} (I + \Delta x F)
\]

where \( F = \begin{pmatrix} 0 & 0 \\ -c & a \end{pmatrix} \) and \( a = E^2(1, 1), c = E^2(2, 1) \). Hence

\[
((I + \Delta x E^2)^{-1} \tilde{G})^l = \left( \frac{1}{1 + a \Delta x} \right)^l (\tilde{G} + \Delta x F \tilde{G})^l.
\]
Then from Lemma 4.2, we know that
\[ \|(\bar{G} + \Delta x F \bar{G})^l\|_2 \leq C_1 C_2 e^{l \Delta x C_1 C_2 \|F\|_2}. \]

Therefore,
\[ \|(I + \Delta x E^2)^{-1} \bar{G})^l\|_2 \leq \frac{1}{|1 + a \Delta x|^l} C_1 C_2 e^{l \Delta x C_1 C_2 \|F\|_2}, \]

for any \( l \), where \( \|F\|_2 = (|E^2(1, 1)|^2 + |E^2(2, 1)|^2)^{\frac{1}{2}} \). And hence
\[ \|\hat{V}_{n+1}\|_2 \leq \frac{C_1 C_2}{|1 + a \Delta x|^{n+1}} e^{C_1 C_2 \|F\|_2 \Delta x} \|\hat{V}_0\|_2 \]
\[ + \Delta x \frac{C_1 C_2}{|1 + a \Delta x|^{n+2}} e^{C_1 C_2 \|F\|_2 \Delta x} - 1 \|(I + \Delta x E^2)^{-1}\|_2 \|E^3\|_2. \]

This finishes the proof. \qed

A.2 Proof of Theorem 4.4

Proof. As in the proof of Theorem 4.3, we first consider the stability for the following discretized system

\[ W_{i+1,j} = \frac{1}{2}(W_{i,j+1} + W_{i,j-1}) - \frac{\Delta x}{2 \Delta y} A_{i,j} \hat{W}_{i,j}(W_{i,j+1} - W_{i,j-1}) - \Delta x E_{i,j} \hat{p}_x, \]

where \( W_{i,j} \) is the approximated values of \( w = (\ln \tilde{\mu}, \hat{\tilde{p}})^T \). This is the discretized system before taking the exponential. This scheme is similar to the central difference scheme except that its coefficient matrices contain the unknown. Apply the Von Neumann stability analysis by freezing the coefficients in \( E^1 \), \( E^2 \) in \( x \) and \( y \) and the unknown \( \tilde{\mu} \) in \( y \), and let

\[ \hat{W}_{i+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-kj^2} W_{i+1,j} \]
where $k = \sqrt{-1}$. Then we can obtain

$$\hat{W}_{i+1}(\xi) = \tilde{G}^{\tilde{\mu}_i}(\xi) \hat{W}_i(\xi) - \Delta x \tilde{E}^{\tilde{\mu}_i} e^{-kj\xi}$$

where we have approximated the unknown $\tilde{\mu}_{i,j}$ in the coefficient matrices by $\tilde{\mu}_i$, for all $j$, e.g. the average value of $\tilde{\mu}_{i,j}$ over $j$, which is equivalent to setting $\xi = 0$ in the Fourier transform of $\tilde{\mu}_{i,j}$. And hence,

$$\tilde{G}^{\tilde{\mu}_i}(\xi) = \frac{1}{2} \left( I - \frac{\Delta x}{\Delta y} A^{\tilde{\mu}_i} \right) e^{k\xi} + \frac{1}{2} \left( I + \frac{\Delta x}{\Delta y} A^{\tilde{\mu}_i} \right) e^{-k\xi}.$$

After $n + 1$ iterations,

$$\hat{W}_{n+1} = \tilde{G}^{\tilde{\mu}_n}(\xi) \hat{W}_n - \Delta x \tilde{E}^{\tilde{\mu}_n} e^{-kj\xi}$$

$$= \tilde{G}^{\tilde{\mu}_n}(\xi) \left( \tilde{G}^{\tilde{\mu}_{n-1}}(\xi) \hat{W}_{n-1} - \Delta x \tilde{E}^{\tilde{\mu}_{n-1}} e^{-kj\xi} \right) - \Delta x \tilde{E}^{\tilde{\mu}_n} e^{-kj\xi}$$

$$= \tilde{G}^{\tilde{\mu}_n}(\xi) \tilde{G}^{\tilde{\mu}_{n-1}}(\xi) \left( \tilde{G}^{\tilde{\mu}_{n-2}}(\xi) \hat{W}_{n-2} - \Delta x \tilde{E}^{\tilde{\mu}_{n-2}} e^{-kj\xi} \right)$$

$$- \Delta x \left( \tilde{G}^{\tilde{\mu}_n}(\xi) \tilde{E}^{\tilde{\mu}_{n-1}} + \tilde{E}^{\tilde{\mu}_n} \right) e^{-kj\xi}$$

$$= \cdots$$

$$= \prod_{m=0}^{n} \tilde{G}^{\tilde{\mu}_m}(\xi) \hat{W}_0 - \Delta x \left( \prod_{m=1}^{n} \tilde{G}^{\tilde{\mu}_m}(\xi) \tilde{E}^{\tilde{\mu}_0} + \prod_{m=2}^{n} \tilde{G}^{\tilde{\mu}_m}(\xi) \tilde{E}^{\tilde{\mu}_1} + \cdots + \tilde{G}^{\tilde{\mu}_n}(\xi) \tilde{E}^{\tilde{\mu}_{n-1}} + \tilde{E}^{\tilde{\mu}_n} \right) e^{-kj\xi},$$

implying that

$$\|\hat{W}_{n+1}\|_2 \leq \| \prod_{m=0}^{n} \tilde{G}^{\tilde{\mu}_m}(\xi) \|_2 \| \hat{W}_0 \|_2 + \Delta x \left( \| \prod_{m=1}^{n} \tilde{G}^{\tilde{\mu}_m}(\xi) \|_2 + \| \prod_{m=2}^{n} \tilde{G}^{\tilde{\mu}_m}(\xi) \|_2 \\
+ \cdots + \| \tilde{G}^{\tilde{\mu}_n}(\xi) \|_2 + 1 \right) \max_m \| \tilde{E}^{\tilde{\mu}_m} \|_2.$$
for any \( l \).

Multiplying \( \bar{G}^{\tilde{\mu}} \) on the left by \( S^{\tilde{\mu}} \) and on the right by \( (S^{\tilde{\mu}})^{-1} \), where as before \( S^{\tilde{\mu}} \) is the eigenvector matrix of \( A^{\tilde{\mu}} \), we get

\[
\bar{H} = \frac{1}{2}(e^{k\xi} + e^{-k\xi})I - \frac{\Delta x}{2\Delta y}(e^{k\xi} - e^{-k\xi})\Lambda.
\]

From Lemma 4.1, we know that if \( \Delta x/\Delta y \max(|\lambda^1|, |\lambda^2|) \leq 1 \), then \( \|\bar{H}\|_2 \leq 1 \), and hence \( \|\bar{H}\|_{2-l+1} \leq 1 \) for any \( l \). Therefore,

\[
\| \prod_{m=l}^{n} \bar{G}^{\tilde{\mu}}_m(x) \|_2 = \| \prod_{m=l}^{n} ((S^{\tilde{\mu}}_m)^{-1} \bar{H} S^{\tilde{\mu}}_m) \|_2 \leq \| (S^{\tilde{\mu}})^{-1} \|_2 \| S^{\tilde{\mu}} \|_2 \| \bar{H} \|_{2-l+1} \prod_{m=l}^{n-1} \| S^{\tilde{\mu}}_m (S^{\tilde{\mu}}_{m+1})^{-1} \|_2 \leq \bar{C}_1 \bar{C}_2 e^\gamma
\]

where \( \bar{C}_1, \bar{C}_2 \) and \( \gamma \) are defined as in Theorem 4.3. Therefore,

\[
\| \hat{W}_{n+1} \|_2 \leq \max(\bar{C}_1 \bar{C}_2, 1) e^\gamma (\| \hat{W}_0 \|_2 + (n + 1) \Delta x \bar{C}) ,
\]

where \( \bar{C} = (\|E^2(1, 1)\| + |E^3(1)|/(2L^{\tilde{\mu}}))^2 + (2|E^2(2, 1)|U^{\tilde{\mu}} + |E^3(2)|)^2)^{1/2} \). Since we set the initial value \( \tilde{\mu}_0 \) to be zero, this means that

\[
\| \ln \tilde{\mu}_{n+1} \|_2 \leq \| \hat{W}_{n+1} \|_2 \leq \max(\bar{C}_1 \bar{C}_2, 1) e^\gamma (\| \ln \tilde{\mu}_0 \|_2 + (n + 1) \Delta x \bar{C}) ,
\]

\[
\| \tilde{\mu}_{n+1} \|_2 \leq \| \hat{W}_{n+1} \|_2 \leq \max(\bar{C}_1 \bar{C}_2, 1) e^\gamma (\| \ln \tilde{\mu}_0 \|_2 + (n + 1) \Delta x \bar{C}) .
\]

Lastly, we show that \( \tilde{\mu}_{n+1,j} \) can also be bounded. From

\[
\ln |a| \leq |\ln a| \Rightarrow |a| \leq e^{\ln |a|}
\]

for any complex number \( a \), and

\[
\sum_{j=1}^{N} | \ln \tilde{\mu}_{n+1,j} | \leq \sqrt{2} \| \ln \tilde{\mu}_{n+1} \|_2 \leq \sqrt{2}(c \| \ln \tilde{\mu}_0 \|_2 + \bar{c}),
\]
where \( c = \max(\bar{C}_1 \bar{C}_2, 1) e^\gamma \) and \( \bar{c} = \max(\bar{C}_1 \bar{C}_2, 1)(n + 1) \Delta x e^\gamma \bar{C} \), we can derive that
\[
\prod_{j=1}^N |\tilde{\mu}_{n+1,j}| \leq \prod_{j=1}^N e^{\|\ln \tilde{\mu}_{n+1,j}\|} = e^{\sum_{j=1}^N \|\ln \tilde{\mu}_{n+1,j}\|} \\
\leq e^{\sqrt{2\bar{c}}} e^{\sqrt{2c} \|\ln \tilde{\mu}_0\|_2}.
\]

So if \( \tilde{\mu}_{0,j} \) are real and satisfy \( \tilde{\mu}_{0,j} \geq 1 \) for all \( j \) and since \( \sum_{j=1}^n a_j^2 \leq \sum_{j=1}^n a_j \) when \( a_j \geq 0 \) for any \( j \), then \( \prod_{j=1}^N |\tilde{\mu}_{n+1,j}| \leq e^{\sqrt{2\bar{c}}} \left( \prod_{j=1}^N \tilde{\mu}_{0,j} \right)^{\sqrt{2c}/N} \). Taking the \( N \)th root of both sides we obtain that the geometric mean of \( \tilde{\mu}_{n+1} \) satisfies
\[
\left( \prod_{j=1}^N |\tilde{\mu}_{n+1,j}| \right)^{1/N} \leq e^{\sqrt{2\bar{c}}/N} \left( \prod_{j=1}^N \tilde{\mu}_{0,j} \right)^{\sqrt{2c}/N}.
\]

This finishes our proof.

\( \square \)

### A.3 Proof of Theorem 4.6

**Proof.** As in the proof of Theorem 4.5, we first establish a stability result for the discretized scheme we obtain prior to our exponentiation step. Again we freeze the value of \( \tilde{\mu} \) to be \( \tilde{\mu}^x \) for each fixed \( x \) and where the \( \tilde{\mu} \) occurs in the coefficient matrices. When we do this, the discretized system before taking the exponential becomes
\[
W_{i+1,j} = \frac{1}{2}(W_{i,j+1} + W_{i,j-1}) - \frac{\Delta x}{2\Delta y} \tilde{A}^{\tilde{\mu}_x} (W_{i,j+1} - W_{i,j-1}) - \Delta x \tilde{E}^{\tilde{\mu}_x},
\]
where again \( W_{i,j} \) is the approximated value of \( w = (\ln \tilde{\mu}, \tilde{p})^T \) at \((i\Delta x, j\Delta y)\), while
\[
\tilde{A}^{\tilde{\mu}_x} = \begin{pmatrix}
E^1(1, 1) & E^1(1, 2)(1 + \epsilon \tilde{\mu}_x)/\tilde{\mu}_x^x \\
E^1(2, 1)\tilde{\mu}_x^x/(1 + \epsilon \tilde{\mu}_x^x) & E^1(2, 2)
\end{pmatrix},
\]
\[
\tilde{E}^{\tilde{\mu}_x} = \begin{pmatrix}
E^2(1, 1) + E^3(1)/\tilde{\mu}_x^x \\
(E^2(2, 1)\tilde{\mu}_x^x + E^3(2))/(1 + \epsilon \tilde{\mu}_x^x)
\end{pmatrix}.
\]
Apply the Von Neumann stability analysis and let

$$\hat{W}_{i+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-kj\xi} W_{i+1,j}$$

where $k = \sqrt{-1}$. Then we can obtain

$$\hat{W}_{i+1}(\xi) = \hat{\tilde{\mu}}^x_i G(\xi) \hat{W}_i(\xi) - \Delta x \bar{E}^x_i e^{-kj\xi}$$

where

$$\tilde{\mu}^x_i G(\xi) = \frac{1}{2} \left( I - \frac{\Delta x}{\Delta y} \bar{A}^x_i \right) e^{k\xi} + \frac{1}{2} \left( I + \frac{\Delta x}{\Delta y} \bar{A}^x_i \right) e^{-k\xi} .$$

After $n + 1$ iterations,

$$\hat{W}_{n+1} = \hat{\tilde{\mu}}^x_n G(\xi) \hat{W}_n - \Delta x \bar{E}^x_n e^{-kj\xi}$$

$$= \hat{\tilde{\mu}}^x_n G(\xi) \left( \hat{\tilde{\mu}}^x_{n-1} G(\xi) \hat{W}_{n-1} - \Delta x \bar{E}^x_{n-1} e^{-kj\xi} \right) - \Delta x \bar{E}^x_n e^{-kj\xi}$$

$$= \hat{\tilde{\mu}}^x_n G(\xi) \left( \hat{\tilde{\mu}}^x_{n-2} G(\xi) \hat{W}_{n-2} - \Delta x \bar{E}^x_{n-2} e^{-kj\xi} \right)$$

$$= \cdots$$

$$= \prod_{m=0}^{n} \hat{\tilde{\mu}}^x_m G(\xi) \hat{W}_0 - \Delta x \left( \prod_{m=1}^{n} \hat{\tilde{\mu}}^x_m G(\xi) \bar{E}^x_m + \prod_{m=2}^{n} \hat{\tilde{\mu}}^x_m G(\xi) \bar{E}^x_m + \cdots \right) \bar{E}^x_n e^{-kj\xi} ,$$

implying that

$$\|\hat{W}_{n+1}\|_2 \leq \|\prod_{m=0}^{n} \hat{\tilde{\mu}}^x_m G(\xi)\|_2 \|\hat{W}_0\|_2 + \Delta x \left( \|\prod_{m=1}^{n} \hat{\tilde{\mu}}^x_m G(\xi)\|_2 + \|\prod_{m=2}^{n} \hat{\tilde{\mu}}^x_m G(\xi)\|_2 \right)$$

$$+ \cdots + \|G\|_2 + 1 \right) \max_m \|\bar{E}^x_m\|_2 .$$
In order to establish the stability results, we need to find non-negative constants $\bar{K}$ and $\bar{\alpha}$ such that

$$\| \prod_{m=l}^{n} \tilde{G}^m (\xi) \|_2 \leq \bar{K} e^{\bar{\alpha}(n-l+1)\Delta x}$$

for any $l$.

From the proof of Theorem 4.5 we know that the eigenvector matrix $\bar{S}^{\tilde{x}}_i$ of $\bar{A}^{\tilde{x}}_i$ and its inverse matrix $(\bar{S}^{\tilde{x}}_i)^{-1}$ satisfy

$$\| \bar{S}^{\tilde{x}}_i \|_2 \leq \| S^{\tilde{x}}_i \|_2 + \epsilon \max(|S(1,1)|, 4|S(2,2)U_\tilde{\mu}|) + \tilde{c}\epsilon^2,$$

$$\| (\bar{S}^{\tilde{x}}_i)^{-1} \|_2 \leq \| (S^{\tilde{x}}_i)^{-1} \|_2 + \epsilon \max(|S(1,1)|, 4|S(2,2)U_\tilde{\mu}|)/|d| + \tilde{c}\epsilon^2/|d|,$$

and

$$\prod_{m=l}^{n-1} \| S^{\tilde{x}}_m (S^{\tilde{x}}_{m+1})^{-1} \|_2 \leq \prod_{m=l}^{n-1} \| I + \frac{\tilde{\mu}^{x}_{m+1} - \tilde{\mu}^{x}_{m}}{d} \bar{D}_{m} \|_2$$

$$\leq \prod_{m=l}^{n-1} (1 + \| \frac{\tilde{\mu}^{x}_{m+1} - \tilde{\mu}^{x}_{m}}{d} \bar{D}_{m} \|_2)$$

$$\leq \prod_{m=l}^{n-1} e^{\| \frac{\tilde{\mu}^{x}_{m+1} - \tilde{\mu}^{x}_{m}}{d} \bar{D}_{m} \|_2}$$

$$\leq e^{\sum_{m=l}^{n-1} \| \frac{\tilde{\mu}^{x}_{m+1} - \tilde{\mu}^{x}_{m}}{d} \bar{D}_{m} \|_2} \leq e^{\tilde{\gamma}}$$

where $\tilde{c} = 8|S(2,2)U_\tilde{\mu}^3|$, $\tilde{\gamma} = TV(\tilde{\mu}) \left( \|D_{m}\|_2 + \frac{\epsilon}{|S(1,1)S(2,2)D_{\tilde{\mu}}|} + |\tilde{c}|c^2 \right)/|d|$.}

Multiplying $\tilde{G}$ on the left by $\bar{S}^{\tilde{x}}_i$ and on the right by $(\bar{S}^{\tilde{x}}_i)^{-1}$, we get

$$\tilde{H} = \bar{S}^{\tilde{x}}_i \tilde{G} (\bar{S}^{\tilde{x}}_i)^{-1} = \frac{1}{2}(e^{k\xi} + e^{-k\xi})I - \frac{\Delta x}{2\Delta y} (e^{k\xi} - e^{-k\xi})\Lambda.$$

From Lemma 4.1, we know that if $\Delta x/\Delta y \max(|\lambda^1|, |\lambda^2|) \leq 1$, then $\| H \|_2 \leq 1$, and
hence \( \| \tilde{H} \|_2^{n-l+1} \leq 1 \) for any \( l \). Therefore,

\[
\| \prod_{m=l}^{n} \tilde{G}^{\mu_m}(\xi) \|_2 = \| \prod_{m=l}^{n} \left( \tilde{S}^{\mu_m} \right)^{-1} \tilde{H} \tilde{S}^{\mu_m} \|_2 \\
\leq \| \left( \tilde{S}^{\mu_l} \right)^{-1} \|_2 \| \tilde{S}^{\mu_l} \|_2 \prod_{m=l}^{n-1} \| \tilde{S}^{\mu_m} (\tilde{S}^{\mu_{m+1}})^{-1} \|_2.
\]

And hence, \( \| \tilde{W}_{n+1} \|_2 \leq c_\epsilon \| \tilde{W}_0 \|_2 + \bar{c}_\epsilon \), where \( c_\epsilon \) and \( \bar{c}_\epsilon \) are given as in Theorem 4.5.

Since we set the initial value \( \tilde{\lambda}_0 \) to be zero, this means that

\[
\| \ln \tilde{\mu}_{n+1} \|_2 \leq c_\epsilon \| \ln \tilde{\mu}_0 \|_2 + \bar{c}_\epsilon, \quad \| \tilde{\lambda}_{n+1} \|_2 \leq c_\epsilon \| \ln \tilde{\mu}_0 \|_2 + \bar{c}_\epsilon.
\]

Lastly, we show that \( \tilde{\mu}_{n+1,j} \) can also be bounded. From

\[
\ln |a| \leq |\ln a| \Rightarrow |a| \leq e^{\ln a}
\]

for any complex number \( a \), and

\[
\sum_{j=1}^{N} |\ln \tilde{\mu}_{n+1,j}| \leq \sqrt{2} \| \ln \tilde{\mu}_{n+1} \|_2 \leq \sqrt{2}(c_\epsilon \| \ln \tilde{\mu}_0 \|_2 + \bar{c}_\epsilon),
\]

we can derive that

\[
\prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \leq \prod_{j=1}^{N} e^{\ln \tilde{\mu}_{n+1,j}} = e^{\sum_{j=1}^{N} |\ln \tilde{\mu}_{n+1,j}|} \\
\leq e^{\sqrt{2}c_\epsilon} \cdot e^{\sqrt{2}c_\epsilon \| \ln \tilde{\mu}_0 \|_2}.
\]

So if \( \tilde{\mu}_{0,j} \) are real and satisfy \( \tilde{\mu}_{0,j} \geq 1 \) for all \( j \), then

\[
\prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \leq e^{\sqrt{2}c_\epsilon} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2}c_\epsilon}.
\]
Taking the $N$th root of both sides we obtain that the geometric mean of $\tilde{\mu}_{n+1}$ satisfies
\[
\left( \prod_{j=1}^{N} |\tilde{\mu}_{n+1,j}| \right)^{1/N} \leq e^{\sqrt{2}c_{\epsilon}/N} \left( \prod_{j=1}^{N} \tilde{\mu}_{0,j} \right)^{\sqrt{2}c_{\epsilon}/N}.
\]
This finishes our proof. \qed