THE ROLE OF SENSING INFORMATION IN PURSUIT-EVASION GAMES

By

Nikhil Karnad

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Dr. Volkan Isler, Thesis Adviser

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ABSTRACT

Today mobile sensor networks find widespread use in environmental monitoring, health care applications, home automation, traffic control and search-and-rescue missions. One can envision robots in civilian environments helping people solve tasks that require specialized sensors or are simply too dangerous for people to risk taking a chance. A pursuit-evasion game is a game played between a pursuer and an evader. The pursuer's objective is to capture the evader, while the evader in turn tries to avoid capture. For example, in a search-and-rescue setting, the search team might be looking for a lost hiker whose actions are unknown. A good pursuit strategy will help the team find the hiker regardless of his movements. Pursuit-evasion games provide a framework in which to study these problems and formulate solutions that provide provable guarantees in the face of uncertainty.

Throughout literature, it is common to assume that the searchers can gather complete information about the location of the evader. However, this formulation does not lend itself to robotics applications where mobile robots have limited sensing capabilities. In this thesis, we study the effect of reducing the sensing information available to the pursuer on the outcome of two well-known pursuit-evasion games.

In the first part, we study a discrete game played on a finite, undirected graph in which the players move from vertex-to-vertex. It is known that there exist graphs, called *cop-win graphs*, on which a pursuer (cop) with complete information captures the evader (robber) in a number of steps of the order of the number of vertices in the graph. However, when the sensing range of the cop is reduced to a neighborhood of just k vertices around him, we show that the time needed to capture the evader grows to an order exponential in the number of vertices in the graph.

The second part examines the well-known Lion and Man game, in which a lion and man play a pursuit-evasion game in a continuous arena. In previous work, it has been shown that there exists a pursuit strategy for the lion to catch the man in the positive quadrant, given certain initial conditions. However, that study relies on the lion knowing the exact location of the man before making its move. We reduce the

sensing ability of the lion to bearing-only information, motivated by mobile robots with monocular vision systems used in tracking applications. We prove that with a bearing-only sensor, the lion can get to within a capture distance equal to the step-size of the game.

The larger objective that motivates the study of both of these problems is the need to answer two important questions in robotic sensor network applications. First, can we use sensors of lower cost to achieve the same results as resources whose costs are difficult to justify? Next, given the sensing uncertainties of current technology in robotics systems, what kind of guarantees can we provide for robots executing tasks in civilian environments. In this thesis, we tackle both of these questions at two levels: we provide sound theoretical results with guarantees on the running times of pursuit-evasion strategies, and, we discuss some of issues that arise when translating these algorithms to real-world implementations.

1. INTRODUCTION

Today, the domain of pursuit-evasion games fuses knowledge from game theory, calculus, control theory and geometric algorithms, finding relevance far from the military techniques that it originated from. In the field of robotics, one finds diverse applications, including collision-avoidance [15], air-traffic control [6], surveillance [15], search-and-rescue [9, 25], health care and environment monitoring, and communication networks.

A pursuit-evasion game involves two types of players: a pursuer and an evader. The pursuer's goal is to capture the evader, while the evader's goal in turn is to avoid capture. The pursuer is said to win the game if he can capture the evader in finite time. In the event that the evader avoids being caught indefinitely, we award the game to the evader. The variations that define a pursuit-evasion game are the number of players, the information that each player can gather during the course of the game (about each other as well as the environment), the restrictions on the motion of each player, the definition of what is a good criterion to optimize as the game is being played (resources such as time and energy are of widespread importance), and specific tasks to which the pursuit-evasion framework is being applied. Our goal is to focus on the role of sensing information on the outcome of the game. More precisely, we study how a reduction in the information available to the players affects the outcome of the game and the capture time of the game.

From a robotics perspective, one can formulate many sensor network tasks as pursuit-evasion games. For example, consider an urban search-and-rescue scenario. When a hazardous environment, such as a building on fire or unstable structures from earthquakes, need to be scanned by a robot to find people trapped inside, the search team cannot make assumptions on what the actions of the people might be. Pursuit-evasion games allow us to model such unpredictable elements as evaders, thus providing worst-case guarantees that work irrespective of the survivor's actions. An example of a good pursuit strategy in this case would be for a robot with appropriate sensors to scan the hallways and rooms inside the building to find a human in the least time possible. Consider a separate task: the use of robots in health care to monitor the elderly. The goal might be to report to a doctor the progress of the individual on a periodic basis, or to raise an alarm in case an accident occurs. Another task of relevance to our Robotics Sensor Networks laboratory at Rensselaer is maintaining connectivity in mobile wireless networks [28]. In general, pursuit-evasion games provide us with a framework in which to study these problems and establish guarantees for pursuer and evader strategies, even in the event of uncertainty in sensing information.

The thesis is organized in three main parts. We start with a survey of the literature on pursuit-evasion games by explaining the origins of the problems at hand and giving an account of related work. For a detailed account we direct the interested reader to Chapter 2.

We then proceed to study the first of our two games, *the cops-and-robbers game*. This pursuit-evasion game is played on a graph, in which the players occupy the nodes and undirected edges represent the possible choices for a player to move from one location to another. One can think of an urban environment modeled as a graph, with the hallways and rooms as nodes and the doors that interconnect them as edges. This version of the game has been called *discrete* pursuit-evasion because the players are restricted to move on the nodes of a graph. In each round, the players move from vertex-to-vertex, with the cop winning the game if he occupies the same node as the robber in finite time. Throughout literature, it is either assumed that the cop knows the exact vertex that the robber occupies at all times, or that the cop is blind and has to rely on random walks. We formulate the sensing model of the cop as k*-visibility*, which means that the cop knows the exact location of the robber only if there is a path of length at most k edges that connect the two players. Outside of this region, the cop does not know where the robber is located. We show that on a well known class of graphs, called cop-win graphs, a reduction in the sensing ability of the cop pushes the capture time of the game from being linear in the number of nodes of the graph to an exponential quantity. This is an instance of a game where the reduction in sensing information does not affect the outcome of the game, but the capture time is greatly increased. These results appeared in [18].

The second problem of interest is a variant of the well-known *lion-and-man game*. In contrast to the cops-and-robbers game, this game is played in a continuous arena. Although the original game was a continuous time analysis, it should be noted that our variant proceeds in rounds i.e. time is discrete. Previous work in the positive quadrant has detailed a lion's strategy that captures the man, but requires the exact location of the man to be known to the lion at all time. We study the effect of reducing the lion's visibility from complete information to bearing-only information, motivated by mobile robots with monocular vision systems. In the circular arena and the positive quadrant, we show that the lion with a bearing-only sensor can get to within a step-size distance of the man. For the positive quadrant game, we show that when the capture distance is reduced to zero, the man wins the game by exploiting the lion's uncertainty in estimating his location. Preliminary results appeared in [19].

The thesis is organized in four chapters as follows. Chapter 2 surveys the literature on pursuit-evasion games in both the discrete setting and the continuous setting, the difference being how the arena is modeled. In Chapter 3, we study the first of the two problems with the cop's sensing reduced to k-visibility instead of global visibility. The continuous-space, discrete-time lion-and-man game with the bearing-only sensing limitation is explained in Chapter 4. We summarize our our contributions and throw light on future work by concluding in Chapter 5.

2. RELATED WORK

In this chapter, we present an overview of previous work done on two specific pursuit-evasion games that relate to applications in mobile sensor networks: the cop-robber game and the lion-man game. The organization of this chapter highlights a fundamental aspect of pursuit-evasion games – the model used to represent the arena in which the game is played.

2.1 Discrete pursuit-evasion: The cops and robbers game

One way to model the environment of a pursuit-evasion game is to use the mathematical concept of graphs. In this formulation, the players each occupy a vertex of a finite graph. Undirected edges connecting the vertices denote possible moves that the players can make from one vertex to another. Parsons [24] described the game as the search for a fugitive with arbitrary speed hiding in a network of tunnels. In that work, we come across the definition of *search number*, which is the minimum number of guards required to catch the fugitive. Megiddo et al. [21] proved that computing the search number of a graph is NP-hard.

In early work, we come across different names for graph-based pursuit-evasion games, for example the hamstrung squad car game and the homicidal chauffeur game, both introduced by Isaacs [15] and the game of cops and robbers [23, 26]. We study the cops and robbers game, played in discrete time on a finite, undirected graph. It is easy to see that if the robber has no visibility (that is, if it cannot gather information about the position of the cops), a single cop can capture the robber on any graph using a simple random-walk strategy. This gives an expected capture time of $O(nm^2)$ where n denotes the number of vertices and m denotes the number of edges in the graph. In [3], Aleliunas et al. presented a strategy for capturing the evader with additional restrictions on the space requirements for the pursuer strategy. Recently, Adler et al. revisited the game and showed that a single pursuer (*hunter*) can catch the evader (*rabbit*) in $O(n \log n)$ time [1]. The strategy works even if the evader can jump from its current vertex to an arbitrary vertex.

It was also shown that this analysis is tight: there are graphs and matching evader strategies which guarantee that no pursuer strategy can capture the evader in less than $\Omega(n \log n)$ steps.

The full visibility version (where the players know each others positions at all times) has received significant attention [23, 26, 8]. It is known that under the full-visibility model, the class of graphs on which a single pursuer suffices is the class of *dismantlable* graphs. The number of pursuers necessary to capture the evader on a graph G is known as the cop number of G. Aigner and Fromme showed that the cop number of planar graphs is at most 3 [2]. In [13], an analysis of the lengths of games on chordal graphs was presented. Further, in [14], Hahn et al provide an algorithmic characterization of reflexive cop-win digraphs when the cop-number k is fixed. The cop number of general graphs is open (c.f. [22, 10, 2]). The problem of determining whether k cops with given initial locations can capture a robber on a given undirected graph is EXPTIME-complete [11]. For the complexity of pursuit in directed graphs, see [11] and references therein.

Isler et al. studied the case where the evader has local visibility [17]. They study a variant where the players move simultaneously, and introduce the notion of i-visibility where a player with i-visibility can see another player only if the distance between them is at most i. It was shown that when the evader has 1-visibility (i.e. can see only the neighbors of its current location), two cops with 1-visibility can capture the evader with high probability on any graph. The expected capture time with two pursuers is polynomial in the number of vertices. A characterization of cop-win graphs where a single pursuer suffices to capture the evader was also presented. It was also shown that when the evader has 2-visibility, the number of cops required becomes unbounded: there are graphs which require $\tilde{\Omega}(\sqrt{n})$ cops to capture an evader with 2-visibility.

2.2 Continuous pursuit-evasion: The lion and man game

There are numerous versions of pursuit-evasion games in continuous arenas. In this section, we focus on only the lion-and-man game and present related work.

The lion-and-man problem was originally posed by Rado in 1925 as follows

> *A lion and a man in a closed arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal?*

The first solution to this problem was generally accepted by 1950, as explained by Littlewood [20] – the lion moves to the center of the arena and then remains on the radius that passes through the man's position. Since they have the same speed, the lion can remain on the radius and simultaneously move toward the man. It turns out that the temporal aspect of the game is crucial in determining the outcome. First, consider the discrete-time version, where at every time step, the players move in turns. For this version, the solution clearly guarantees capture[1]. However, Besicovitch showed that when the game takes place in continuous time, the man wins! While the lion can get arbitrarily close to the man in finite time, capturing the man takes forever. For the continuous-time version, Alonso et al [5] presented an almost-optimal strategy by showing that the lion can get within a distance c of the man in time $O(\frac{r}{s}\log\frac{r}{c})$ where r is the radius of the arena and s is the maximum speed of the players.

Sgall studied a variant of the lion-and-man game which takes place in the non-negative quadrant of the plane [27]. This version of the game first appeared in [12]. The outcome of the game depends on the initial positions of the players. Let (x_p, y_p) and (x_e, y_e) be the initial position of the pursuer and the evader respectively. If either $x_e \geq x_p$ or $y_e \geq y_p$, it is easy to see that the evader wins. Sgall showed that in the remaining case, the pursuer wins. He presented an almost optimal strategy that is quadratic in the pursuer's distance from the origin and the slope of the line connecting the player's initial locations.

Recently, Isler et al. showed that the lion can capture the man inside any simply-connected polygon [16]. Alexander et al. presented a sufficient condition

[1]In this game, the lion is said to capture the man if the distance between them is zero.

for a natural greedy strategy to succeed in arbitrary dimensions [4]. More recently, Bopardikar et al. [7] initiated the study of sensing limitations in the lion-and-man game. In their model, the lion can observe the man's location only if the distance between the players is less than a given threshold. In this thesis, we focus on a different type of sensing limitation and study pursuit strategies for a pursuer equipped with a bearing-only sensor.

3. k-VISIBILITY PURSUIT-EVASION ON A GRAPH

One way to model the environment of a pursuit-evasion game is to use the mathematical concept of graphs. In this formulation, attributed to Parsons [24], the players each occupy a vertex of a finite graph. Undirected edges connecting the vertices denote possible moves that the players can make from one vertex to another. Throughout literature, this game has been referred to as the *cops and robbers game*. The cop is said to win the game if within a finite number of steps he occupies the same vertex as the robber i.e. the players are coincident. In the event that the robber avoids capture indefinitely, he is said to have won the game. The outcome of the game is thus either a *cop-win* or a *robber-win*.

3.1 Our results

A graph G is called cop-win if a single cop can capture the robber on G. In this paper, we study the effect of reducing the cop's visibility on cop-win graphs. In particular, we study what happens if the cop can see the robber only if the distance between the two is less than or equal to some threshold value. Throughout the paper, we study connected, undirected graphs with self-loops (i.e. the players can stay in their vertices if they choose to). In the first part of the paper, we focus on the worst-case scenario, and assume that the robber has global visibility. Hence, it can see the cop at all times.

First, we obtain a positive result and show that a cop with small or no visibility can capture the evader on any cop-win graph (even if the robber still has global visibility). The cop uses a randomized strategy. We show that the capture probability within a finite time interval is non-zero. This yields an upper bound on the expected capture time which is exponential in the number of vertices (Section 3.3).

On the negative side, we obtain a lower bound and show that the increase in capture time is indeed exponential. We show that there exists a cop-win environment and a robber strategy such that the expected capture time for any cop strategy is exponential in the number of vertices (Section 3.4). More precisely, we show that for any n, there exists an environment with $O(n)$ vertices for which the expected capture time is $\Omega(3^n)$.

3.2 Game model

We study the cops and robbers game on undirected graphs. Throughout the paper, we focus on the single cop version. The locations of the players are specified at the beginning of the game. The players move in turns. A move is considered as a transition from a vertex to any of its adjacent vertices. At each time step, first the evader moves along an edge. Next the pursuer moves. For a graph $G = (V, E)$, we use the following definition of neighborhood of a vertex $v \in V$:

$$N(v) = \{v\} \cup \{u|(v, u) \in E\}$$

Note that the neighborhood is closed, i.e. the vertex v is included in the neighborhood of itself. This means that the players can stay in their current vertices if they choose to. We use the notation $d(u, v)$ to denote the length of a shortest path (distance) between u and v. The cop captures the evader if he can move onto the evader's current vertex.

We say the pursuer has k-visibility, when the location of the evader is revealed to the pursuer only if the distance between them is at most k. A pursuer with global visibility can see the evader at all times. An evader with k-visibility is defined similarly.

3.3 Upper bound on capture time

Recall that a graph G is called cop-win if a global visibility pursuer can capture the evader on G. In this section, we show that a pursuer with no visibility can also capture the evader on a cop-win graph. The class of cop-win graphs admits a simple characterization:

Definition 3.3.1. (Distmantlable Graphs [26, 8]). *Suppose i and j are nodes of a graph H such that $N(i) \subseteq N(j)$. The map that takes i to j and every other vertex of H onto itself is a homomorphism from H to $H - \{i\}$. This operation is* *called a fold of graph H and we say vertex i folded onto vertex j. A finite graph H is said to be dismantlable if there exists a sequence of folds reducing H to a graph with one vertex.*

A graph G is cop-win if and only if G is dismantlable [26, 8].

Definition 3.3.2. (Folding Tree). *Let $G = (V, E)$ be a dismantlable graph. Given a vertex $v \in V$, a folding tree for G with respect to v is a tree rooted at v, which represents a folding sequence of G where v is the only remaining vertex. When a vertex i folds onto j in the sequence, j becomes the parent of i in the folding tree.*

We proceed as follows. First, we obtain an upper bound on the capture time for a pursuer with global visibility.

Lemma 3.3.1. *Let G be a dismantlable graph with n vertices. There exists a vertex v such that the cop with global visibility can start from v and capture the evader in at most n steps.*

In [17], it was shown that the pursuer can start at the root of a folding tree and chase the evader in such a way that (i) the pursuer remains an ancestor of the evader on the folding tree, and (ii) every time the evader revisits a leaf-to-root path P, the pursuer's height on P decreases. These two results combined imply that the pursuer never revisits a vertex during the game. This, in turn, implies Lemma 3.3.1.

Next, using this upper bound, we obtain a randomized (mixed) strategy for a pursuer with no visibility.

Theorem 3.3.1. *The expected time it takes for a pursuer with no visibility to capture an evader with global visibility on a dismantlable graph $G = (V, E)$ is at most $2n\Delta^n$, where Δ is the maximum degree of G, and $n = |V|$.*

Proof. By Lemma 3.3.1, a pursuer with global visibility can start from the root of the folding tree and capture the evader in at most n steps. Which means that, no matter how the evader moves, there exists a sequence of pursuer moves, of length at most n, which guarantees capture. The pursuer with no visibility will travel to the root and then guess the sequence of evader moves. Since there are at most Δ

possibilities at each step, the probability of a correct guess is at least $\frac{1}{\Delta^n}$. Hence, by repeating this process with independent guesses, the pursuer is expected to capture the evader in Δ^n trials. Each trial contains a trip to the root followed by a sequence of n moves. Since the length of each trial is at most $2n$, the expected capture time is at most $2n\Delta^n$. □

3.4 A lower bound on the capture time

Theorem 3.3.1 suggests the possibility of an exponential increase in expected capture time due to loss of visibility. In this section, we show that there exists a class of graphs and evader strategies on these graphs such that the expected capture time for any pursuer strategy is lower bounded by a quantity that is indeed exponential in the number of vertices.

We start with the case where the pursuer has no visibility. Later, we will extend the result to the case where the pursuer has k-visibility for arbitrary k. First we introduce the environment for which we will present an evader strategy.

3.4.1 Definitions

Given $n \in \mathbb{N}$, we construct a graph G_n as follows (Figure 3.1).

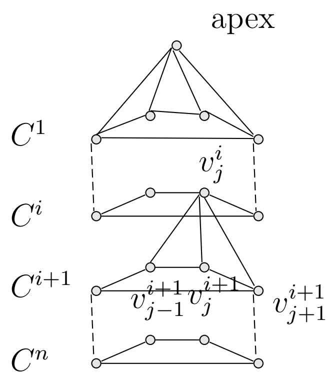


Figure 3.1: Construction of the graph G_n, the environment on which our pursuit-evasion game is played.

Start with a cycle of length 4. Make n copies $C^1, C^2, \ldots, C^n$ and "stack" them one below the other where C^i is placed above C^{i+1}. We align the cycles so that for each vertex $v \in C^i$ there is a unique vertex in C^{i+1} which is directly below v. We can now rename the vertices in $C^i = \{v_1^i, \ldots, v_4^i\}$ such that v_j^i is *above* v_j^{i+1}. For each v_j^i, $i = 1, \ldots, n-1$, $j = 1, \ldots, 4$, we introduce edges to $v_{j-1}^{i+1}, v_j^{i+1}, v_{j+1}^{i+1}$ (We define $v_5^{i+1} = v_1^{i+1}$ and $v_0^{i+1} = v_4^{i+1}$.).

Finally, we add a special vertex to this graph and connect it to all vertices in C_1. We call this vertex the *apex* of our graph.

Definition 3.4.1. The environment *refers to G_n for a given n.*

Given two cycles C^i and C^j, we say that C^i is above (below) C^j if $i < j$ (respectively $i > j$). Similarly, we say v_k^i is above v_k^j if $i < j$. We say v_k^i is directly above v_k^j if $i = j - 1$. The players are *vertically aligned* if one player is on a vertex that is above the other player's vertex.

It should be noted that the environment is dismantlable and hence cop-win.

A *configuration* describes the state of the game by specifying the vertices that the pursuer and evader occupy.

Definition 3.4.2. A configuration *is given by the pair (p, e) where p is pursuer's current vertex and e is evader's current vertex.*

Recall that the evader is captured if at any point in time, the pursuer and the evader both occupy the same vertex of the graph.

We define $T[p, e]$, the *capture time* for a given configuration (p, e), as the number of moves required for the pursuer located at p to capture the evader located at e. Without loss of generality, we assume that the pursuer minimizes the capture time whereas the evader maximizes it. The capture time for a graph $G = (V, E)$ is given by $\max_{p,e \in V} T[p, e]$. In what follows, we obtain a lower bound on the capture time of G_n. First, let us take a closer look at the global visibility game and show the unique pursuer strategy to capture the evader in this case.

3.4.2 Capture on G_n with a global visibility pursuer

In this section, we show how an optimal strategy for a pursuer with global visibility can capture an evader with global visibility on G_n. The intuitive idea for this pursuer strategy stems from the fact that if the players are not vertically aligned at any point in the game, then the evader resets the progress made by the pursuer thus far by escaping to the top level.

Consider a configuration (p, e). There are three possibilities.

Case (i) The pursuer is at or below the evader's level. The evader avoids capture by staying on his level while moving away from the pursuer horizontally.

Case (ii.a) The pursuer is above the evader and the players are *vertically aligned*. Irrespective of the evader move, the pursuer can move down and maintain vertical alignment.

Case (ii.b) The pursuer is above the evader and the players are not vertically aligned. The evader can *reset* the game by exploiting the horizontal separation between the players and moving up diagonally. The only way for the pursuer to establish horizontal alignment is to move up to the apex.

Thus, an optimal pursuit strategy would avoid configurations in Case (i) and (ii.b) until capture. This yields the following theorem.

Theorem 3.4.1. *In an optimal pursuer strategy, the pursuer remains above the evader until capture. Moreover, there exists an optimal evader strategy where the capture occurs on C^n.*

In the next section, we obtain a lower bound on the expected capture time with a pursuer with zero-visibility.

3.4.3 Evader Strategy

In this section, we describe an evader strategy S against a pursuer with no visibility. We will then show that the expected capture time of any pursuer strategy on G_n against the S is large.

Recall that to capture the evader, the pursuer must remain above it (Theorem 3.4.1). Since the pursuer with zero-visibility has no way of knowing where the evader is at the end of each round, he simply has to make a guess to maintain this invariant. The evader strategy S then penalizes each wrong guess by resetting the game.

The evader strategy S randomizes how the evader moves down the stack by picking, with equal probability, one of three neighboring vertices at the lower level.

To capture the evader, the pursuer will have to make n *consecutive* "correct" guesses.

The crucial component of S is to penalize a single "wrong" guess by resetting the entire game. Suppose the pursuer p is one level above the evader e. The evader picks at random one of the three vertices that are neighbors of e, located on the level below his current level. These always exist, except if the evader is at C_n. Observe that with probability 2/3, the evader and pursuer are no longer vertically aligned. If this happens, the evader can reset the game by moving to the top level (Section 3.4.2, Case (ii) b).

With the remaining probability of 1/3, the pursuer is still vertically aligned with the evader and the evader has got closer to C_n, resulting in a "correct" guess for the pursuer. To capture the evader, in the worst case, the pursuer needs n such consecutive guesses.

Next, we obtain a lower bound on the expected capture time.

The state of the game is measured by the capture time i.e. the expected number of steps required for the pursuer to capture the evader. The sequence of pursuer guesses and game outcomes can be modeled as a random walk on the directed graph shown in Figure 3.2. Here, the probability of progress (a "correct" guess by the pursuer) toward the capture state is 1/3 and the probability of the game being completely reset (a "wrong" guess by the pursuer) is 2/3.

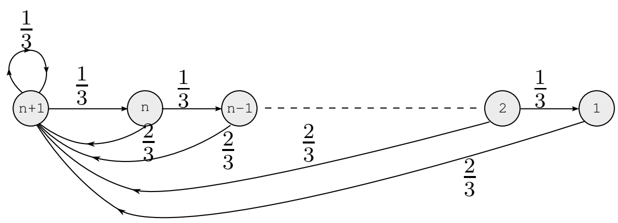


Figure 3.2: A random walk on a line: The pursuer has to make consecutive correct guesses to prevent the game from being reset at any point in time.

The expected time it takes for this process to start at the left endpoint of the chain and arrive at the right endpoint is $\Omega(3^n)$ which gives us a lower bound on the expected duration of the game.

Theorem 3.4.2. *For every $n \in \mathbb{N}$, there exists an environment G_n with $4n + 1$*

vertices and an evader strategy S such that the expected capture time of any pursuer strategy (for the zero-visibility pursuer) against S is $\Omega(3^n)$.

3.4.4 Generalization to the k-visibility pursuer

We can obtain a lower bound on the capture time of a k-visibility pursuer using the same environment and the evader strategy S: suppose the pursuer starts at the apex and the evader is located at a vertex in C^{k+1}. Therefore, the pursuer can not see the evader. The evader follows S and resets the game if the pursuer makes a wrong guess – i.e. if the pursuer is no longer above the evader. Note that this requires entering pursuer's vision range. However, the pursuer still has to move to the apex to reach a configuration where he is above the evader (the evader will stop at C^{k+1}). On the other hand, if the pursuer reaches C^{n-k} by making correct guesses, the evader will be visible from that point on. Therefore, the pursuer does not need to make any further guesses.

To analyze the capture time, we can use a random walk on the line similar to the one shown in Figure 3.2. Since the pursuer stops guessing after reaching C^{n-k}, the length of the path is $n - k$ which gives us the following lower bound.

Theorem 3.4.3. *For every $n \in \mathbb{N}$, there exists an environment G_n with $4n + 1$ vertices and an evader strategy S such that the expected capture time of any pursuer strategy (for a k-visibility pursuer) against S is $\Omega(3^{n-k})$.*

Therefore, unless k is comparable to n, increasing the pursuer's visibility does not help much against a global visibility evader.

3.5 Symmetric Visibility

In this section, we study the cops and robbers game when both players have k-visibility. The case when $k = 0$ has been studied in [1] and $k = 1$ has been studied in [17]. In this section, we first show that the pursuer can establish eye-contact with the evader on any graph under the symmetric visibility model. That is, the pursuer can force the game into a configuration (p, e) such that p and e are visible from each other.

When studying the symmetric visibility game, the following question arises:

What is the class of graphs on which the evader can be captured with a single pursuer? The answer clearly depends on k. As mentioned earlier, when $k = 0$ the evader can be captured on any graph [1]. Further, the results in the previous sections imply that the evader can be caught on any distmantlable graph for arbitrary k. However, it is worth noting that it is possible to have situations where the evader will enter and leave the pursuer's sight until capture (an example is the game described in Section 3.4.4). It has also been shown that the class of graphs where the evader can be caught for $k = 1$ is larger than the class of dismantlable graphs [17].

In this section, we present a general result on establishing eye-contact in the symmetric visibility game. Next, we present a characterization of environments where a class of greedy algorithms suffices for capture (Section 3.6). On these environments, the pursuer can first establish eye contact and then follow the greedy strategy to capture the evader. Hence, this characterization yields a sufficient condition for capture in the symmetric visibility case. However, due to their widespread use [29, 7, 4], we believe that a characterization of environments where the greedy strategy works will be of independent interest as well.

We first show that the pursuer can establish eye-contact with the evader on any graph G for any k with high probability. The pursuer uses the following *strategy* S_p: Let n be the order of the graph G. The pursuer strategy consists of rounds of length n. At the beginning of the round, the pursuer is at some vertex which may be known by the evader. The pursuer picks a vertex (which we refer to as a destination) uniformly at random, moves to the destination via the shortest path, and waits there until the end of the round.

The basic idea is that, no matter what strategy the evader follows, the evader will be visible from a vertex v at the end of the round and the pursuer picks this vertex with probability $\frac{1}{n}$. The main technical challenge is the following. The pursuer picks v at the beginning of the round. If the evader can infer pursuer's choice using observations during the round, it may avoid being seen by avoiding the neighborhood of v. The following lemma shows that this can not happen. Namely, the evader can not obtain useful observation without risking being found by the pursuer.

Lemma 3.5.1. *On any graph connected G, and for any $k \geq 0$, a pursuer with k-visibility can move in such a way that, with high probability, no matter how an evader with k visibility moves, the game reaches a configuration (p, e) where e is visible from p.*

Proof. Suppose, for contradiction, that there exists an evader strategy S_e which guarantees that the probability of pursuer's finding the evader in any finite time interval is zero. Since the evader will not see the pursuer during the execution of S_e, the information it has about possible locations of the pursuer will be the same, regardless of the pursuer's strategy. Now, suppose the pursuer executes the strategy S_p described above and let π_e denote the evader's path while it is executing S_e during the round. Note that this is a fixed path in the graph and the evader will follow this path unless it is seen. Let w be the last vertex of π_e. With probability at least $\frac{1}{n}$, the pursuer will pick a neighbor of w as a destination and hence, find the evader. This contradicts the existence of S_e. □

The expected time to find the evader using S_p is $O(n^2)$. It can be improved to $O(n \log n)$ by replacing S_p with the pursuer strategy in [1].

Note that for $k = 0$ the lemma implies capture. For $k = 1$, the result can be inferred from some of the results in [17].

3.6 Greedy strategies

Let (p, e) be the current configuration of a cops and robbers game on a graph G. The evader moves to a vertex $e' \in N(e)$ and suppose e' is visible from p. A natural pursuit strategy is to move toward e' on *a* shortest path from p to e'. Variants of such greedy strategies are extensively used in robotics because they are easy to implement and typically require little bookkeeping [29, 7, 4].

Two remarks are in order. First, observe that greedy strategies do not guarantee capture on some cop-win graphs. For example, in the environment of Section 3.4 (Figure 3.1), if the players are on the same cycle, a greedy pursuer will keep looping forever on this cycle. In contrast, an optimal strategy would move to the apex (which requires increasing the distance for some time) and then capture the evader.

Second, even though the greedy strategy is straightforward in Euclidean environments, it is not well-defined on graphs when there are multiple shortest paths. Let p and e be the vertices of the pursuer and evader respectively. Let $p_1, \ldots, p_q$ be neighbors of p that minimize the distance from e. Which vertex should the pursuer select? In this paper, we study two variants:

Class A Greedy Pursuit: In this version, we focus on a worst-case scenario and assume that the *evader* picks the shortest path the pursuer has to follow. The significance of this model is that if the pursuer can win in this model, then *any* choice of p_i yields capture.

Class B Greedy Pursuit: In this version, the *pursuer* is free to pick any shortest path between p and e'. If the pursuer can win in this model, this means that there is *some* choice of p_i that yields capture but additional reasoning (or perhaps randomization) is needed to find the right greedy strategy.

It is easy to see that if Class A pursuit succeeds, so will Class B pursuit. Also, note that in some environments (e.g. trees, Euclidean environments), the shortest path is unique. Hence, the distinction between Class A and B strategies disappear. Finally, there are environments where Class A pursuit fails but Class B pursuit succeeds. An example is shown in Figure 3.3.

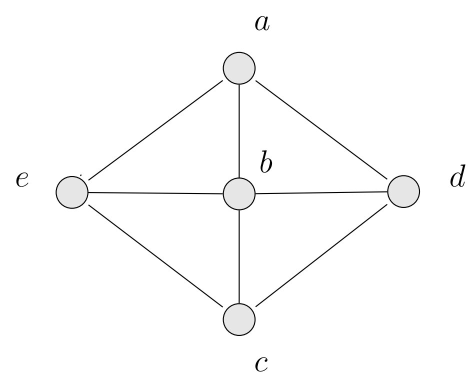


Figure 3.3: On this graph Class A greedy pursuit fails but Class B greedy pursuit succeeds.

In Figure 3.3, suppose the pursuer is at a and the evader is at b. In Class A

pursuit, the evader can move to c and force the pursuer to move to e. Afterward, the evader can make the pursuer chase the evader forever on the cycle $e \to c \to d \to a$. In contrast, a class B greedy pursuer can move to b and capture the evader.

3.6.1 Class A Greedy Pursuit

In this section, we use study the following greedy strategy. Let p and e be the locations of the pursuer and the evader when it is the pursuer's turn to move. Let $P = \{\pi_1, \dots, \pi_k\}$ be the set of all shortest paths between p and e. Since the pursuer's choice of which shortest path to follow is arbitrary and we focus on the worst case scenario, we assume that the evader picks a path $\pi \in P$ from this set and the pursuer moves toward the evader on π.

We now present an algorithmic characterization (Algorithm Mark-Greedy) of graphs on which Class A greedy pursuit succeeds. In the following, $N(u, v)$ denotes the neighborhood of u with respect to v: $N(u, v) = \{w \in N(u) : d(w, v) = d(u, v) - 1\}$.

Theorem 3.6.1. *A graph G is greedy-cop-win if and only if Algorithm Mark-Greedy(G) marks all configurations of G.*

Proof. Suppose that all configurations are marked. We show, by induction on the order of marked configurations, that the greedy pursuit strategy guarantees capture. The first configuration marked is of the form (v, v) for some vertex v. Therefore, if the game ever reaches this configuration, the evader is captured by definition. For inductive hypothesis, assume that greedy pursuit guarantees capture if the game ever enters one of the first k marked configurations. Consider the configuration (p, e) marked $(k + 1)^{th}$. It must be that, no matter which vertex e' the evader moves to, and no matter which shortest path it chooses for the pursuer, the pursuer will enter a configuration which was marked in the first k steps and win the game afterward.

Now suppose that after the execution of Mark-Greedy some configurations were unmarked. Let (p, e) be such a configuration and suppose the game starts at this configuration. It must be that there exists a vertex e' that the evader can move to, and a shortest path π between p and e' such that the resulting configuration

(p', e') after the pursuer moves to p' on π is unmarked (Otherwise (p, e) would be marked.). Hence, the evader can guarantee that the game always remains in unmarked configurations. All capture configurations are of the form (v, v) and hence, marked. Therefore, the pursuer can never capture the evader with the greedy strategy. □

Algorithm Mark-Greedy(Graph $G = (V, E)$):

/* Initially, all configurations are unmarked */

Mark all configurations (v, v) for every vertex $v \in V$.

Repeat

Mark (p, e) if for all $e' \in N(e)$ and for all $p' \in N(p, e')$, (p', e') is marked.

Until no further marking is possible.

3.6.2 Class B Greedy Pursuit

Recall that the distinction between Class A and Class B pursuit is due to the choice of the shortest path the pursuer will follow. In Class B pursuit, the pursuer is free to choose any shortest path. Algorithm Mark-Greedy can be modified to recognize Class B greedy-cop-win graphs. The only modification needed is to replace the body of the loop with the line: Mark (p, e) if for all $e' \in N(e)$, *there exists* $p' \in N(p, e')$ such that (p', e') is marked. The proof of correctness is similar to Theorem 3.6.1 and omitted.

In addition to an algorithmic characterization, we show that Class B greedy pursuit succeeds on chordal graphs in the next section. However, there are non-chordal graphs where Class B pursuit succeeds as well. For example, the graph in Figure 3.3 has a chord-less cycle ($ecda$).

3.6.2.1 Chordal graphs are Class B greedy-cop-win

A chord of a cycle is an edge connecting two non-consecutive vertices of a cycle. A graph G is chordal (or triangulated) if every cycle of length four or more contains a chord.

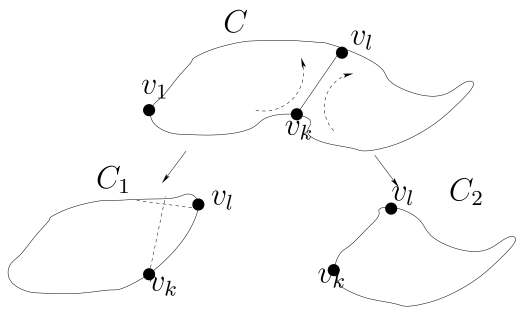
Chordal graphs form an important class of graphs which includes trees, cliques and interval graphs. In this section, we show that Class B greedy strategy works on chordal graphs.

Definition 3.6.1. *Let $C = \{v_1, \ldots, v_k\}$ be a cycle of a chordal graph G. Three vertices v_i, v_{i+1} and v_{i+2} form an ear of C if (v_i, v_{i+2}) is an edge.*

We utilize the following lemma.

Lemma 3.6.1. *Every cycle C of a chordal graph has an ear.*

Since an ear is also a chord, Lemma 3.6.1 yields: A graph G is chordal if and only if every cycle of G has an ear.



Proof.

Figure 3.4: Every cycle C in a chordal graph has an ear.

Let $C = v_1, \ldots, v_n$. We prove the lemma by induction on n. For the base case ($n = 3$), the lemma is trivially true. For the inductive step, assume that every cycle of length up to n in a chordal graph has an ear. Consider a cycle C with $|C| = n$. Since G is chordal, C must have a chord (v_k, v_l) for some k and l. Therefore we can divide C into two cycles C_1 and C_2. To obtain C_1, we start from v_k, follow the edge (v_k, v_l) and then C counter-clockwise. To obtain C_2, we follow C clockwise after v_l. We can rename the vertices so that $C_1 = \{v_1, \ldots v_k, v_l, v_{l+1}, \ldots, v_n\}$. By the inductive hypothesis, both C_1 and C_2 have ears. We focus on C_1. Two possible cases arise. In the first case, the ear is either $e_1 = (v_{k-1}, v_k, v_l)$ or $e_2 = (v_k, v_l, v_{l+1})$. If this is not the case, the ear of C_1 is also an ear in C and we are done. Therefore we focus on the first case. We form a new cycle C_1' as follows. If e_1 is the ear, $C_1' = \{v_1, \ldots v_{k-1}, v_l, \ldots, v_n\}$ (i.e. C_1' is obtained by deleting v_k from C_1). Otherwise, if e_2 is the ear, $C_1' = \{v_1, \ldots v_k, v_{l+1}, \ldots, v_n\}$ (i.e. v_l is deleted from C). We now apply the same argument to C_1'. Through this process, either we get to a point where C_1' has an ear which does not involve the chord or $|C_1'| = 3$, in which case, it will also have an ear which does not involve the chord.

□

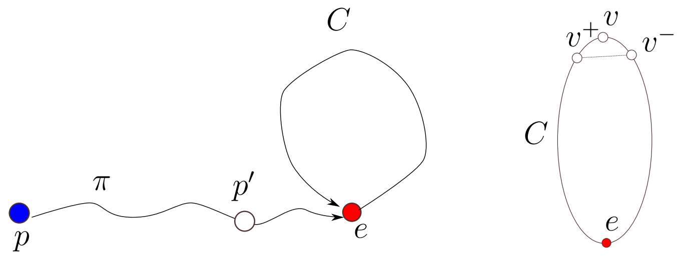


Figure 3.5: The two cases that arise in Lemma 3.6.2.

Lemma 3.6.2. *Let G be a chordal graph. By following the Class B greedy strategy described above, the pursuer can capture the evader on G.*

Proof. Suppose the pursuer and evader play the cops and robbers game on a graph G, starting from an arbitrary configuration. Let $d_1, d_2, \ldots, d_i, \ldots$ be a sequence where d_i is the distance between the players after the pursuer's i^{th} move. First, observe that when the pursuer is following the greedy strategy, no matter which shortest path is chosen, the distance between the pursuer and the evader after the pursuer's move never increases. We claim that if G is chordal, the pursuer can select the shortest paths in such a way that after a finite number of steps, the distance between the players decreases. Clearly, this implies capture. Let k be the current distance between the pursuer and the evader. For contradiction, assume that the claim is false. No matter what strategy the pursuer uses, the evader must revisit a vertex. Let e be the first revisited vertex and suppose the evader revisits e after l steps for the first time. Let C be the cycle formed by the evader's trajectory. Let p be the location of the pursuer when the evader is at e initially. Let π be a shortest path of length k between π and e. Note that, by assumption, the distance between the players remains k throughout these l steps. First suppose $k > l$. This case is shown in Figure 3.5-left. In this case, remaining on π is a valid Class B greedy strategy as it guarantees a separation of k. But then, when the evader revisits e, the distance between the players will decrease, leading to a contradiction. Therefore it must be that $k \leq l$. In this case, the pursuer can stay on π until e and follow C afterward. This also maintains a separation of k and hence it is a valid Class B strategy. Since the evader travels the full cycle C, it must go through a vertex v which, together with its neighbors v^- and v^+, form an ear (Figure 3.5-right). Let $H = (\pi + C) \subseteq G$. After the evader moves to v^+ and the pursuer completes his move on H, the distance between the two on H is k. However, since (v^-, v^+) is an edge, the distance between them on G is at most $k-1$. This means that the pursuer is not following a shortest path which is a contradiction. □

3.6.3 Symmetric visibility and greedy strategies

Consider the following model for symmetric visibility cops and robbers game. Both players have k-visibility, for some k. Suppose the game configuration is (p, e) with $d(p, e) = k$. Imagine the evader moves to e' with $d(p, e') = k+1$. If we make the assumption that the pursuer can instantaneously observe the evader's motion and infer that the evader is on e', then it is easy to see that on greedy-cop-win graphs, the pursuer can capture the evader. First, it finds the evader (Lemma 3.5.1). Since the greedy pursuit guarantees that the distance never increases after the pursuer's move, the evader will be visible until the end of the game. If, however, the pursuer cannot observe the evader's motion instantaneously, the pursuer may not follow the greedy strategy. Recent results for greedy pursuit in convex planar environments under this model can be found in [7].

4. BEARING-ONLY PURSUIT

In this chapter, we study a well-known pursuit-evasion game called the lion (pursuer) and man (evader) game [20, 5, 27] under the sensing limitation that the lion has a bearing-only sensor, like a monocular camera. Thus, the pursuer can observe a ray that contains the evader but can not measure the location of the evader on this ray. In other words, the pursuer does not know the position of the evader exactly but has an estimate that corresponds to a line segment containing the evader. The main question we seek to answer is whether the pursuer can capture the evader under this sensing limitation.

There are various versions of the lion and man game depending on the assumptions made on the motion model of the players and on the environment the game is played in. For details on the literature, we direct the interested reader to Chapter 2.

4.1 Our results

We study the discrete-time continuous-space motion model, where the players move in turns and have identical maximum velocities. We assume that the lion is equipped with a bearing-only sensor and cannot sense the position of the man exactly. With such a sensor, a natural greedy pursuit strategy is to move toward the man along the sensing ray.

We first show that, in a circular arena, the lion can reduce its distance to the man to one in $O(R^4)$ steps using the greedy strategy. Our main result involves the version of the game that takes place in the first quadrant. In this version, the greedy pursuit fails (Figure 4.1). The lion can not execute Sgall's strategy, as this strategy requires knowing the man's exact location *before making a move*. In this thesis, we present a new two-state pursuit strategy which guarantees that the lion can reduce its distance to the man to the step-size. An interesting aspect of our strategy is that it requires the pursuer to combine multiple observations and the knowledge about the evader's maximum speed to compute its next move. We also show that the lion

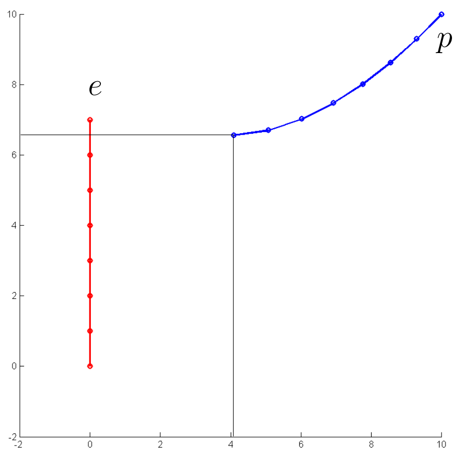


Figure 4.1: The greedy pursuit fails. The man escapes by crossing the lion's coordinate along the Y-axis.

(with a bearing-only sensor) can not reduce the distance between the players to zero with probability one.

In typical robotics applications, the step-size is expected to be small. In particular, if the lion is a disk-shaped robot and the step size is equal its radius, the lion can hit the man with the strategy presented in this paper.

4.2 Game Model

Pursuit-evasion games are defined by: a motion model and sensing model for the players, a notion of capture of the evader by the pursuer, and a model of the environment where the game is played. In this section, we give a formal definition of the lion and man games that we are considering. We study a discrete-time, continuous-space game where the players move in an alternating sequence. The game proceeds in rounds which consists of the following. A sensing phase where the players gather their information. The evader moves. The players sense again, followed by the pursuer's move.

Motion Model: The players are assumed to be holonomic robots, i.e., they can move in any direction instantaneously. Both players are assumed to have the same maximum velocity. The distance that the robots can travel in one time-step is called its *step-size*. Without loss of generality, we assume the maximum step-size to be 1. A time step is considered atomic in the sense that a single control input is applied during the whole length of the time step, i.e., a player can pick a direction and move unit distance along that direction.

Sensing Model: The pursuer is equipped with a bearing-only sensor, i.e., at each sensing phase it can obtain a ray on which the evader lies. The pursuer can compute the exact location of the evader *after* he makes his move, by intersecting consecutive sensing rays. We note that when it is the pursuer's turn to move, he does not know the exact evader position. All the pursuer can do is to construct a line-segment estimate of possible evader locations by intersecting his sensing ray with a unit disc centered at the previous evader location (see Figure 4.2). The evader is assumed to know the exact location of the pursuer (since we are interested in analysing worst-case scenarios).

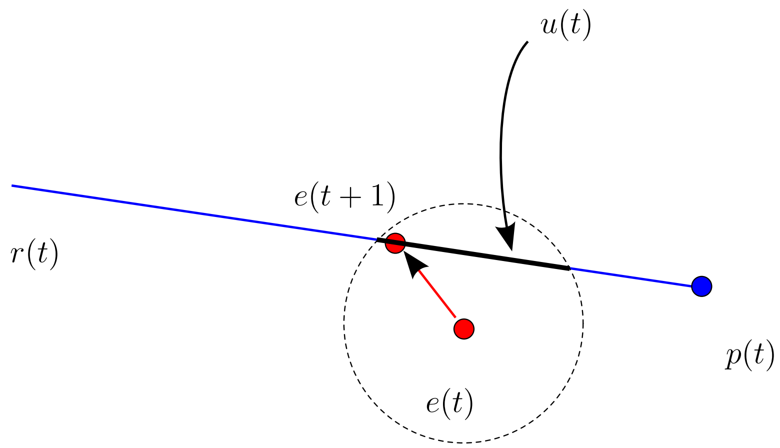


Figure 4.2: The pursuer intersects his sensing ray $r(t)$ with a unit disc centered at $e(t)$ to compute the evader estimate $u(t)$.

Notion of Capture: Let $e(t) \in \mathbb{R}^2$ and $p(t) \in \mathbb{R}^2$ be the positions of the evader and pursuer at time t, respectively. We say that the evader is *captured* by the pursuer if for some $t_f \geq 0$, $||e(t_f) - p(t_f)||_2 \leq c$, where c is constant for a given game, called the *capture distance*. Throughout this paper, we will use $c = 1$. We say that the *pursuer wins* the game if it captures the evader in a finite number of steps, otherwise, the evader wins.

Environment Model: We study the lion and man game in two types of environment: (a) a circular arena of radius R, as considered in the original version of lion and man game (b) the positive quadrant arena, as considered in Sgall [27]. We show in the following subsection that, in a circular arena, using the greedy strategy of moving directly towards the evader, the pursuer can capture the evader.

4.2.1 Bearing-only pursuit in circular arena

Let the radius of the circular arena be R. The pursuer is equipped with a bearing-only sensor and therefore knows the ray on which the evader lies. A natural greedy strategy for the pursuer is to move directly toward the evader along the line joining them.

Lemma 4.2.1. *Using the greedy pursuit strategy in a circular arena of radius R, a pursuer gets to within a capture distance of* 1 *unit from an evader in $O(R^4)$ steps.*

Proof. Let d be the initial distance between the pursuer and the evader. If the pursuer uses the greedy strategy, the distance between the pursuer and evader is non-increasing. The evader can maintain the distance between the players only by moving away from the pursuer along the line joining them. For all of his other moves, the greedy pursuer move reduces the distance between the players. Since the environment is bounded, the evader hits the boundary in a limited number of steps, if it moves along the line joining the pursuer and evader. When the evader reaches the boundary, we show that there is a lower bound on the distance gained by the pursuer in the next step. We show that (a) If the distance between the pursuer and evader is greater than 1 the distance between them decreases by at least $\frac{1}{8R^2}$ just after the evader hits the boundary. (b) The evader either reaches the boundary in $O(R)$ steps or the pursuer gains more than item (a) above if the evader moves so as to not reach the boundary in $O(R)$ steps. From (a) and (b) we conclude that the average progress in one step is $O(1/R^3)$. The initial distance d is less than or equal to $2R$. Therefore the total number of steps taken by the pursuer to capture the evader is $O(R^4)$.

We consider the case when the evader lies on the boundary of the environment at the beginning of the round (See Figure 4.3) . Let P, E be the positions of the

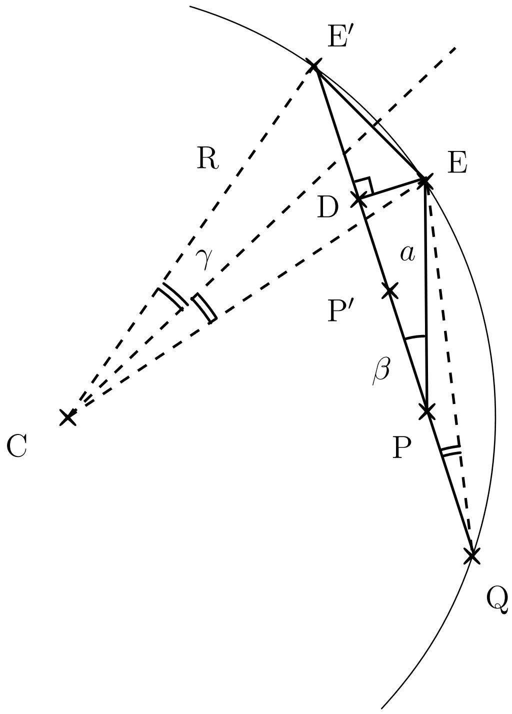


Figure 4.3: A lower bound on distance gained: the evader E moves along the boundary to E' and the pursuer makes a greedy move from P to P'.

pursuer and evader respectively at the beginning of the round and P', E' be their positions at the end. For obtaining a lower bound, we can assume that $E'E = 1$. Moreover, we assume that E' lies on the boundary since the distance PE' is biggest for this case (for any other point, $E'E$ and PE are same but the included angle $\angle PEE'$ is less implying the above statement). Let C be the center of the circular arena and $PE = a$. From $\Delta CEE'$, using $CE = CE' = R$, we have $\sin\gamma = \frac{1}{2R}$. The angle $\angle E'QE = \gamma$, since the angle subtended by the arc EE' at the center of the circle is twice of that subtended at the circumference. The angle $\angle E'PE = \beta \geq \gamma$. We want to lower bound the distance gain $g = PE - P'E'$. We have

$$\begin{aligned} P'E' &= P'D + DE' \\ &= a\cos\beta - 1 + \sqrt{1 - a^2\sin^2\beta} \\ g &= a(1-\cos\beta) + (1 - \sqrt{1 - a^2\sin^2\beta}) \end{aligned}$$

Note that g is a monotonic function of a and $g \to 0$ as $a \to 0$, independent of

β. Both the terms in the expression of g are positive. Therefore,

$$g \geq a\,(1 - \cos\beta) \geq a\,(1 - \cos\gamma) \qquad (\because \beta \geq \gamma)$$
$$\geq a\left(1 - \sqrt{1 - \frac{1}{4R^2}}\right)$$

Writing a completion of squares under the square root i.e. $a^2 - 2ab$ as $a^2 - 2ab + b^2 - b^2$, we get

$$g \geq a\left(1 - \sqrt{1^2 - \left(2 \cdot 1 \cdot \frac{1}{8R^2}\right) + \left(\frac{1}{8R^2}\right)^2 - \left(\frac{1}{8R^2}\right)^2}\right)$$

Dropping the negative quantity $-\left(\frac{1}{8R^2}\right)^2$ from under the square root increases its value, but the negative sign in front of the square root means the overall value of the expression on the right-hand side of the inequality decreases. Therefore,

$$g \geq a\left(1 - \sqrt{1^2 - \left(2 \cdot 1 \cdot \frac{1}{8R^2}\right) + \left(\frac{1}{8R^2}\right)^2}\right)$$
$$\Rightarrow g \geq a\left(1 - \sqrt{\left(1 - \frac{1}{8R^2}\right)^2}\right)$$
$$\Rightarrow g \geq \frac{a}{8R^2}$$

Since $a \geq 1$, the distance gain is at least $O(1/R^2)$ in the round after the evader reaches the boundary.

We now consider the evader to be strictly inside the arena. We want to show that the evader hits the boundary in $O(R)$ steps if it wants to keep the gain less than $\frac{1}{8R^2}$ before it reaches the boundary. As noted before, the evader can keep the total distance gain to 0 if it moves along the line joining the pursuer and evader. However, it would hit the boundary in less than $2R$ steps if it uses this strategy. The distance gain at each step is a function of the angle made by the line of evader move to the line joining pursuer and evader. Let us call this angle of motion θ_i at round i. The gain g at each round as a function of the distance a and θ_i is

$$g = a + 1 - \sqrt{a^2 + 2a\cos\theta_i + 1} \tag{4.1}$$

For $a \geq 1$, g is a monotonically decreasing function of a (we can verify this by checking that $\frac{\partial g}{\partial a} \leq 0$). Moreover g increases as θ_i increases. Substituting $a = 1$ and $\theta_i = \gamma$ in Equation 4.1, we can verify that $g \geq \frac{1}{8R^2}$. Thus, if $\theta_i > \gamma$ during any round, the distance gain is greater than $\frac{1}{8R^2}$. Therefore, we assume that $0 \leq \theta_i \leq \gamma$. Now consider two axes centered at the evader's initial position, one directed along the line joining the pursuer and evader and the other perpendicular to it. Since the evader moves at an angle θ_i to the line joining the pursuer and the evader in each round, the distance moved along the axes are $\cos\theta_i$ and $\sin\theta_i$. To obtain an upper bound on the number of steps required to hit the boundary, we can use $\theta_i = \gamma$. Since the diameter of the arena is $2R$, the number of steps taken to hit the boundary is $\min(\frac{2\text{R}}{\cos\gamma}, \frac{2\text{R}}{\sin\gamma})$. We have $\sin\gamma = O(1/R)$ and therefore $\cos\gamma = O(1)$. Thus the number of steps taken to hit the boundary is $O(R)$ if the evader is restricted to move at an angle less than γ at each step.

□

4.2.2 Bearing-only pursuit in positive quadrant

We proceed to study the more interesting case of a semi-bounded environment, the positive (first) quadrant. A positive result for this environment was proposed in *A solution to David Gale's Lion and Man Problem*, by Sgall [27]. He proved that for any winning pursuer strategy in the positive quadrant, the following invariant must be satisfied [27]. .

Invariant 1. *At the end of each round, the coordinates of the evader should lie between the coordinates of the pursuer and the origin.*

The Invariant 1 is broken if any one of the coordinates of the evader is greater than the corresponding coordinate of the pursuer. In such a case, the evader escapes by moving along the origin parallel to the axis for which its coordinate is greater. Further, Sgall provided a pursuer strategy, henceforth referred to as *Sgall's Lion* *strategy*, which uses complete information of the evader (man) location to capture him. With a bearing-only sensor it is not possible for the pursuer to obtain the exact position of the evader before the pursuer makes his move. The pursuer can only have an estimate of the line segment in which the evader lies. This limitation prevents the pursuer equipped with a bearing-only sensor to apply Sgall's strategy directly.

We show that there exists a pursuer strategy which uses a point on the evader estimate (line-segment $u(t)$ in Fig. 4.2) to make each of its moves, resulting in overall finite capture time for the case when $c \geq 1$. We call this a conservative move, because the pursuer ensures that Invariant 1 is not broken no matter where on the estimate the evader actually lies. We proceed to show that if the capture distance is zero, then the pursuer cannot win the game with probability 1. In the following section, we explain the pursuer's strategy to capture the evader assuming that the pursuer knows the initial position of the evader exactly. Thereafter, we present a correctness proof of the pursuer strategy in Section 4.4 and show that our results hold even if the pursuer doesn't know the initial position of the evader.

4.3 Lion and Man in the First Quadrant

We now turn to the version of the lion and man game that takes place in the first quadrant. Before explaining the bearing-only pursuit strategy for this environment, we give a brief overview of the strategy of a pursuer that can observe the evader's position at all times.

4.3.1 Sgall's Lion strategy

In an earlier paper [27], Sgall presented a pursuer strategy that guarantees capture of the evader in the positive quadrant of the plane, given that the initial conditions satisfy Invariant 1.

Let P and E be the initial locations of the pursuer and the evader respectively. Suppose P and E satisfy Invariant 1. Find a point Q on the line EP such that P lies between Q and E, and a circle C centered at Q, passing through P, touches (or cuts) both the X-axis and the Y-axis. The main idea is for the pursuer to make his moves in such a way that the circle C centered at Q and passing through the pursuer's current location advances further and further away from Q until the evader is trapped. The pursuer executes his move in the following manner.

Suppose the evader moved from E to E' such that $|EE'| \leq 1$ (where 1 is the maximum distance covered in one time-step). The ray QE' intersects a circle centered at P of radius 1 at two points. Of these, the pursuer picks the point farthest from Q and moves to it, call it P^+. This is the end of a round. We will refer to this move as the *Lion's move with respect to Q.*

Sgall showed that the Lion's strategy ensures capture by proving that (a) the square of the distance of P from the center Q increases by 1 unit, irrespective of where the evader moves, and, (b) the pursuer is always inside the line segment connecting the evader to center Q. These two conditions guarantee capture of the evader. Note that the pursuer's distance to Q can be viewed as a measure of *progress*. We adopt this terminology in the rest of the paper.

4.3.2 Bearing-only strategy

In order to execute Sgall's Lion strategy, the pursuer needs to know the exact location of the evader before making his move. Although our pursuer can use the bearing-only ray to triangulate the position of the evader, he will know the exact evader location only *after* he has moved. Therefore, he cannot use this information to execute an exact Lion's move.

The bearing-only pursuit strategy starts by computing a center Q as in Sgall's strategy. The strategy then proceeds in two main phases: (i) employing the original lion strategy with respect to Q *whenever possible* and, if not (ii) guarding the pursuer's progress while "catching up" with the evader in a finite number of steps.

We adapt Sgall's Lion strategy by using a conservative estimate of the evader's location: Let the position of the pursuer at the beginning of a round be $P \in \mathbb{R}^2$ and that of the evader be $E \in \mathbb{R}^2$. After the evader move, the pursuer builds an estimate of the evader position by intersecting his sensing ray with a disc of unit radius centered at E. Call this estimate E^+E^-, where E^- is the end of the estimate closer to Q than E^+. The pursuer assumes that the evader is at E^+ and plays Sgall's

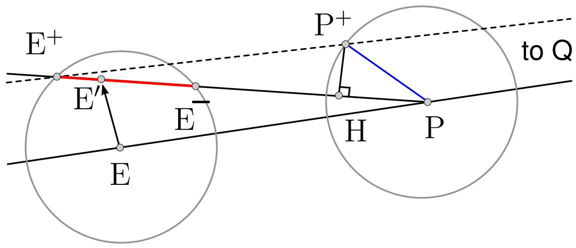


Figure 4.4: Capture condition: If the evader is captured w.r.t the conservative estimate E^+ i.e. whenever $|P^+E^+| \leq 1$, then he is captured no matter where on the estimate he actually is.

Lion strategy. By making this move, the pursuer ensures that Invariant 1 is never broken i.e. the evader does not escape even if the pursuer's guess turns out to be wrong. Therefore, we refer to E^+ as the *conservative estimate.*

The following lemma justifies using the conservative estimate.

Lemma 4.3.1. *Let E^+ be the conservative estimate and P^+ be the pursuer location after the pursuer's move. If $|P^+E^+| \leq 1$, the evader is captured.*

Proof. Suppose the pursuer moved to P^+ (see Figure 4.4) assuming that the evader was at the conservative estimate E^+. Further, suppose that the evader is captured if he is actually at E^+ i.e. $|P^+E^+| \leq 1$. We show that the evader is captured no matter where he actually is on E^+E^-.

Drop the perpendicular from P^+ on to the line PE^+. Since $|PE^-| > 1$ (otherwise the evader would be captured soon after his move), the foot of the perpendicular, call it H, lies inside the circle of radius 1 centered at P. The distance of P^+ from the line PE^+ is least at H and monotonically increases to $|P^+E^-|$ and then $|P^+E^+|$. Therefore

$$|P^+H| < |P^+E^-| < |P^+E^+| \leq 1$$

which proves that for all points on E^+E^-, the evader is within a distance of 1 from P^+, implying capture. □

When the pursuer's guess is wrong, the points Q, P^+ (pursuer's conservative move) and E' (evader's actual location) are not collinear. The evader is now on one side of the line l through Q and the pursuer. Depending on where the evader moves to next, it is possible that the lion's move may not exist for the pursuer. See Figure 4.5 for an example of when the lion's move does not exist.

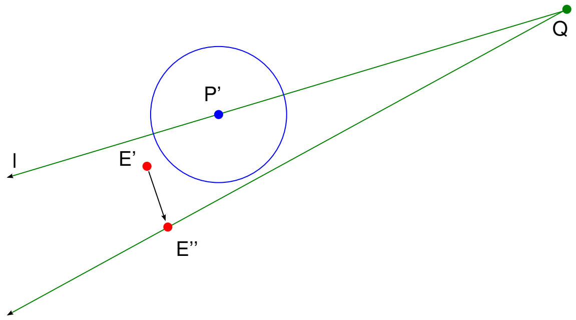


Figure 4.5: When the players are misaligned, the evader can move in such a way that the lion's move does not exist.

Whenever the lion's move exists, the pursuer simply continues his progress. However, when the lion's move ceases to exist, the *guarding phase* is triggered and the pursuer switches to a *guarding strategy*. We now explain how the pursuer executes this guarding strategy.

4.3.3 Guarding Strategy

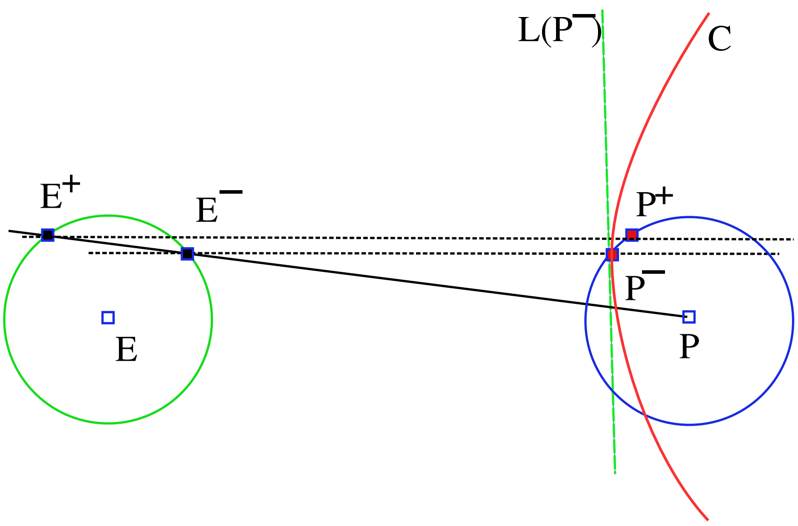


Figure 4.6: The guarding phase of the pursuer's strategy, in which he prevents the evader from crossing the line $L(P^{-})$.

Call the point he should have been at to continue the Lion's strategy as P^{-} (Figure 4.6). Let $L(P^{-})$ be the line through P^{-} tangent to the circle centered at Q

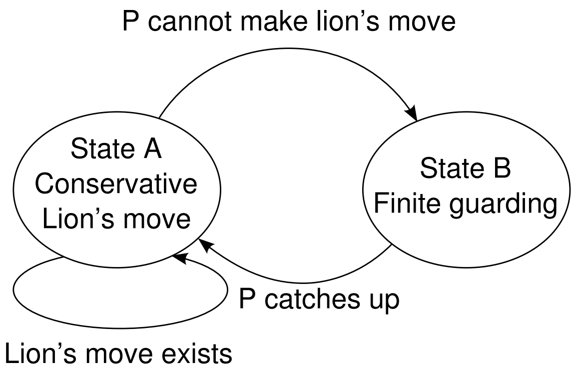


Figure 4.7: Two-state pursuer strategy. The pursuer makes progress whenever he is in State A. When in State B, he guards his previous progress and goes back to State A in a finite number of steps.

passing through P^-. Since the line through E^- and Q is perpendicular to $L(P^-)$, the projection of E^- onto $L(P^-)$ is P^-. If the pursuer were at P^- instead of P^+, he could prevent the evader from crossing $L(P^-)$ just by moving toward the evader's projection on that line. We call this *guarding* the line $L(P^-)$. The guarding strategy involves the pursuer moving from P^+ to P^- and then guarding $L(P^-)$ by staying on $L(P^-)$ and moving toward the evader's projection.

Figure 4.7 illustrates the states in the overall strategy.

In the next section we show that:

1. The guarding strategy preserves the pursuer's progress. In other words, suppose the distance between Q and P^- (the "correct" pursuer location) when the pursuer switches to the guarding mode is d. We will show that when the pursuer returns to the Lion's strategy with respect to Q, his distance to Q will be at least d.

2. In going from the lion's game to the guarding strategy and back to the lion's game, the guarding phase takes a finite number of steps.

In the next section, we will show that this pursuit strategy yields capture.

4.4 Analysis

The pursuer starts by computing the circle center Q and proceeds with the Lion's strategy using the conservative estimate. Suppose the pursuer is at P and the evader at E. The evader moves to a point on E^*E^- and the pursuer moves to the point P^* as described in our conservative Lion's move. Suppose that the evader is not at E^*. Then, the evader, P^* and Q are not collinear. After the next evader move, if the pursuer cannot execute the lion's move (see Figure 4.5 for an example), it triggers the guarding phase of the pursuit strategy.

Lemma 4.4.1. *Suppose, after the pursuer's move, $P' = P^+$, E' and Q are not collinear. Let $|P^-Q| = s$ when this happens There exists a pursuer strategy which guarantees that one of the following happens in a finite number of steps:*

(i) P', E' and Q are collinear and $|P'Q| \geq s$.

(ii) $|P'E'| \leq 1$.

Proof. Let $L(P^-)$ be the tangent at P^- to the circle C centered at Q, passing through P^- (See Figure 4.6). Note that $L(P^-)$ touches both of the axes (since C also does).

Let the radius of C be $s = |QP^-|$. At the beginning of the guarding phase, the pursuer observes the evader's (conservative) move to, say, E^{++}. Let x be the intersection of $L(P^-)$ with the line segment QE^{++}. We refer to x as *the evader's projection onto* $L(P^-)$. If the pursuer reaches the projection (correct guess) in the first step, condition (i) holds and we are done.

Otherwise, the pursuer starts guarding $L(P^-)$ by moving onto the point on $L(P^-)$ that is closest to the evader's projection onto $L(P^-)$. Clearly, the distance between the pursuer's location and the evader's projection is bounded by $\delta = |P^+P^-|$. The guarding phase ends if the pursuer can move to the evader's projection.

During the guarding phase, the evader is inside the area bounded by one or both of the coordinate axes, the ray QP' from the center Q passing through the pursuer's location, and the line $L(P^-)$. As the pursuer guards the line $L(P^-)$, notice that the ray QP' is rotating toward the ray QE'. Thus, before the pursuer hits the axis its moving toward, it is guaranteed that these two rays will cross (unless the evader crosses $L(P^-)$ first). When the rays cross, the pursuer can simply move on to the evader's projection. Since the pursuer has stayed on $L(P^-)$, he is outside circle C centered at Q passing through P^- i.e. after some finite time interval, $|QP'| \geq s$ and the pursuer can resume with the Lion's strategy without loss of progress. Therefore, as long as the evader does not attempt to cross $L(P^-)$, the event described in part (i) of the lemma will happen.

In the remaining case, the evader crosses the line $L(P^-)$.

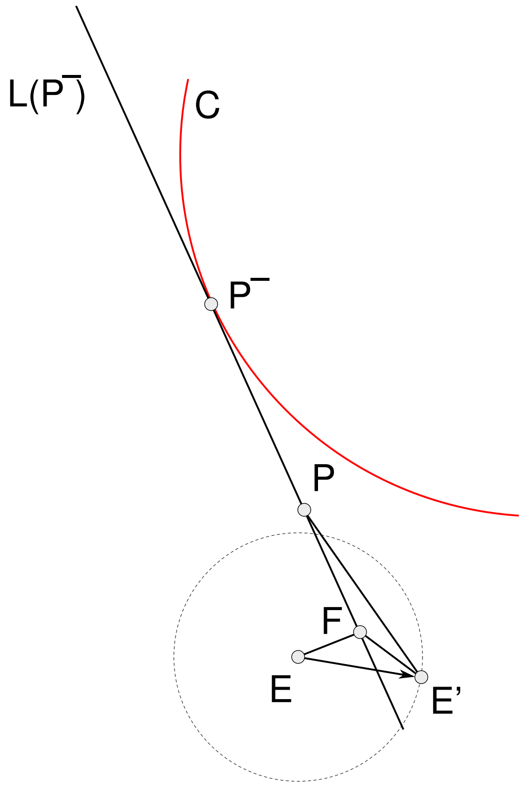


Figure 4.8: If the evader E crosses the line $L(P^-)$, then the pursuer P moves toward E' and the final distance between the players is at most $\delta = |P^+P^-|$.

Figure 4.8 illustrates such a case. Suppose the evader was at E. His projection F on the line $L(P^-)$ is a distance at most $\delta = |P^+P^-|$ away from P i.e. $|PF| \leq \delta$. The evader crosses $L(P^-)$ and lands at E' such that $|EE'| \leq 1$ (maximum step-size). Since the angle $\angle EFE'$ is greater than $\pi/2$ (the evader crossed), and $|EE'| = 1$, we have $|FE'| \leq 1$. Apply triangular inequality in $\Delta FE'P$ and we get

$$|PE'| \leq |PF| + |FE'| \leq \delta + 1$$

Now, the pursuer moves along PE' a unit distance to a point, call it P'. Then $|P'E'| = |PE'| - 1 \leq \delta$. Thus, soon after the evader crosses, the distance between the players is at most δ. □

We now bound δ, the distance between points P^+ and P^-. Let P be the pursuer location before it moved to P^+. By the definition of the lion's move, we have the angle $\angle P^- P P^+ \leq \pi/2$ which means that $\delta \leq \sqrt{2}$. In fact, a tighter bound is possible, given by the next lemma.

Lemma 4.4.2. *In every round, the distance (δ) between the position that the pursuer moves to (P^+) and the position that the pursuer should have moved to (P') is always upper bounded by* 1*.*

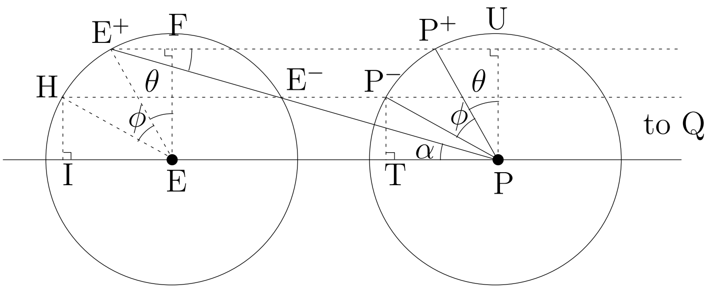


Figure 4.9: The sensing ray intersects the pursuer's circle before the evader's circle. The pursuer estimates the position of the evader in E$^+$E$^-$ with E$^-$ on the evader's circle. The *conservative move* for the pursuer is from P to P$^+$ and the maximum possible error in this conservative move is P$^+$P$^-$. The point Q is assumed to be at infinity (or $z = \infty$) for worst case analysis.

Proof. Let P, E be the positions of pursuer and evader respectively at the beginning of a round, $|PE| = s$, and α be the bearing angle of the evader's position after the evader's move (see Figure 4.9). Let Q (not shown in Figure 4.9) be the center of the circle in Sgall's strategy that lies on the line through E and P to the right of P and $|PQ| = z$. Now, depending on the values of s and α, the sensing ray from P to the evader's position after the evader's move may (a) intersect the pursuer's circle before the evader's circle (Figure 4.9) or (b) intersect the evader's circle before the pursuer's circle (Figure 4.10). For case (a), the evader lies in the segment E^+E^-, and for case (b), the evader is in the segment E^+P^-. The point P' to which the pursuer should move lies in the segment P^+P^- (where P^+ is obtained by the intersection of E^+Q with the unit circle centered at P; P^- is obtained by the intersection of E^-Q

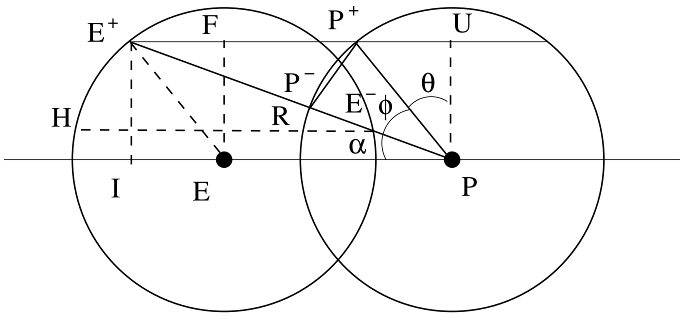


Figure 4.10: The sensing ray intersects the evader's circle before the pursuer's circle. The pursuer estimates the position of the evader in E$^+$P$^-$ with E$^-$ on the pursuer's circle. The *conservative move* for the pursuer is from P to P$^+$ and the maximum possible error in this conservative move is P$^+$P$^-$. The point Q is assumed to be at infinity (or $z = \infty$) for worst case analysis.

or E^+P with the unit circle centered at P depending on case (a) or (b)). Therefore, $\delta \leq |P^+P^-|$ by construction and our problem reduces to proving that $|P^+P^-| \leq 1$.

We note that $|P^+P^-|$ is a function of z, s, and α. For a fixed value of s and α, as the point Q moves away from P, i.e., as z goes to ∞, observe that P^+ rotates about the unit circle centered at P at a rate quicker than P^- does. Thus, $|P^+P^-|$ increases as z increases and for our upper bound proof we can set $z = \infty$ and proceed (the Figures 4.9 and 4.10 are for $z = \infty$). Depending on the case (a) or (b) in the above paragraph, we have the following two results whose proof we defer to the appendix.

1. If the sensing ray intersects the pursuer's circle before the evader's circle (see Figure 4.9), then for $s \geq 1$, $|P^+P^-| \leq 1$ with equality occurring at $s = \sqrt{3}$ and $\alpha = \pi/6$ (Lemma 4.4.5).

2. If the sensing ray intersects the evader's circle before the pursuer's circle, i.e., the point P^- lies in the line segment E^+E^- (see Figure 4.10), then for $1 \leq s \leq 2$, $|P^+P^-| \leq 1$ with equality occurring at $s = \sqrt{3}$ and $\alpha = \pi/6$ (Lemma 4.4.6).

Therefore, for $s \geq 2$, $|P^+P^-| \leq 1$ from result 1 above, since the sensing ray always intersects the pursuer's circle before the evader's circle. For $1 \leq s \leq 2$, the results 1 and 2 together imply that we have $|P^+P^-| \leq 1$ irrespective of whether the sensing ray first intersects the pursuer's or evader's circle. Thus, we have $\delta \leq |P^+P^-| \leq 1$ in every round of the game. □

4.4.1 Arbitrary Capture

For the case when the capture distance $c \geq 1$ (where 1 is the value of the maximum distance that can be covered in one time-step), we have showed that a pursuer with a bearing-only sensor can get to within a distance of c from the evader. In this section, we show that exact capture ($c = 0$) is not possible. For values between 0 and 1 it remains unclear as to which player has a winning strategy.

Lemma 4.4.3. *For a capture distance $c = 0$, there exists an evader strategy that allows him to avoid capture indefinitely (and hence win the game) with arbitrarily high probability.*

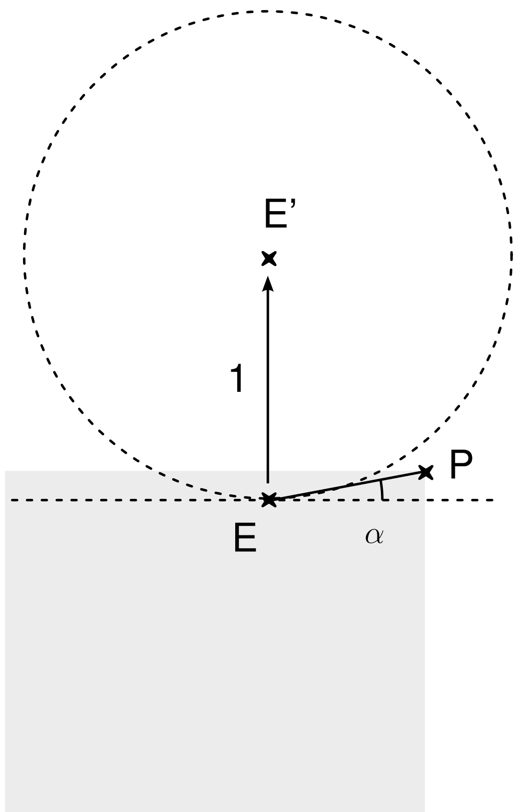


Figure 4.11: Starting configuration of a game in which the evader E wins for capture distance 0. By picking the initial value of α, the evader guarantees that $|PE'| > 1$. The invariant region is shaded.

Proof. We show that there exists an instance of the game with capture distance 0 where the evader has a winning strategy. Consider a game which starts with the pursuer at P and the evader at E. The evader makes his move to some E', such that:

1. $|PE| < 1$

2. Suppose E' is 1 unit vertically above E, then $|PE'| > 1$ (see Figure 4.11).

The evader crosses the pursuer, say vertically (along the Y-axis), by moving from E to E' such that $|EE'| = 1$ (see Figure 4.11). The pursuer's sensing ray lies along PE' and his estimate is the whole of PE' because $|PE| < 1$.

In $\Delta E'EP$, $|PE'| > 1$ by construction, which prevents the pursuer from sweeping along the line PE' as he would end up short of E'. This would break Invariant 1 and the pursuer would lose the game.

Given that the game starts with a configuration where the angle α (see Figure 4.11) is such that $|PE'| > 1$, for all successive pursuer moves that preserve the invariant, the value of α is non-increasing (the parallel pursuer move preserves α and the rest decrease its value). This guarantees that $|PE'| > 1$ for all rounds of the game.

In order to protect against the evader's move to E', the pursuer is left with two options: (i) assume that the evader is at the conservative end of the estimate (E') and play the Lion's strategy, and (ii) preserve the invariant for all points on $|PE'|$, thus not making any progress in the X-direction.

In order to prevent the first option, we provide a randomized evader strategy. Instead of just moving straight up to E', the evader picks between two points on PE' to move to. One of them is the extreme E' (prevents the pursuer sweep strategy) and the other is anywhere along PE', as close as possible to P. (see Figure 4.12) This second move guarantees that if the pursuer executes a Lion's move w.r.t. E' and ends up with X-progress, there exists an evader move that breaks the invariant in the other direction (along the X-axis). These two moves prevent the pursuer from making progress in the X-direction, thus leaving the pursuer with only the second option. A symmetric argument applies for the evader crossing along the other axis.

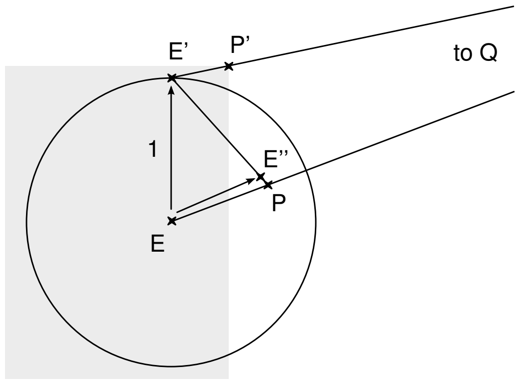


Figure 4.12: The evader picks between E' and E'' randomly. The Lion's move from P to P' w.r.t. E' breaks the invariant (shaded region) if the evader is actually at E''.

By picking the probabilities of each of the two evader moves, we can compute the probability that the evader avoids capture for n steps, for any $n \in \mathbb{N}$. The important observation is that there exists a randomized evader strategy that enables the evader to avoid capture with high probability. □

Theorem 4.4.1. *If the capture threshold is at least one (the maximum distance of a move possible in one time-step), a pursuer with a bearing-only sensor can capture the evader in a finite number of steps by following our two-state bearing-only strategy. Exact capture is not possible as there exists an evader strategy that wins the game with high probability.*

Proof. Whenever the pursuer makes the Lion's move, he makes a definite progress as explained by Sgall in [27]. In our strategy, the pursuer plays a single Lion's game with finite capture time. However, after a pursuer move, the Lion's move may not exist. We showed that whenever this happens, either (i) the pursuer returns to the original Lion's game in a finite number of steps and claims the progress possible with the correct lion's move, or (ii) the distance between the players drops below one and hence, the evader is captured if the capture threshold is at least one.

The whole game can now be viewed as a finite sequence of games $G_1, G_2, G_3, \ldots, G_n$ where games with odd indices are parts of a single Lion's game (with increasing progress) and the even indices are guarding games that last a finite number of steps and preserve the pursuer's progress from one Lion's game to the next.

We now provide an upper-bound on the capture time. Suppose the game starts with the pursuer at (x_0, y_0) and the evader at (x'_0, y'_0) and let $\alpha_0 = (y_0 - y'_0)/(x_0 - x'_0)$ be the initial slope of the line joining them. Then, the total capture time for the Lion's game (sum of the times for $G_1, G_3, G_5, \ldots$) as derived in [27] is:

$$\begin{aligned} T_L &= max\{(x_0 + y_0(\alpha_0 + \sqrt{1 + {\alpha_0}^2}))^2, \\ &\quad (y_0 + x_0({\alpha_0}^{-1} + \sqrt{1 + {\alpha_0}^{-2}}))^2 \end{aligned}$$

A single guarding game lasts for a time of at most the maximum of the X and Y coordinates of the pursuer at P^*, which, by construction of the initial circle centered Q and the imposition of Invariant 1, is bounded from above by the maximum of the X and Y coordinates of the center Q. Thus the capture time for a single guarding game is given by

$$\begin{aligned} T_G &= max\{x_0 + y_0(\alpha_0 + \sqrt{1 + {\alpha_0}^2}), \\ &\quad x_0 + x_0({\alpha_0}^{-2}(1 + \sqrt{1 + {\alpha_0}^2}))\} \end{aligned}$$

Since, in the worst case, the switch from the Lion's game to a guarding game happens at the end of each time step, the total capture time is bounded by $T_L T_G$.

When the capture distance is zero, we provide an evader strategy that wins the game with high probability. Intuitively, this suggests that for arbitrary capture $(0 < c < 1)$, it is likely that the evader can avoid capture, however, the investigation of whether there exists a winning strategy for either player remains open. □

4.4.2 Knowledge about the evader's initial location

We started the paper with the assumption that the pursuer knows the exact initial location of the evader. In this section, we remove this assumption. The main idea is to have the pursuer perform a "safe" initial move and obtain the evader's position by triangulation. This is formalized in the following lemma.

Lemma 4.4.4. *Call the two-state pursuer strategy described earlier as S_p. As explained in Section 4.3.2, S_p requires the initial location of the evader for our analysis to hold. If the pursuer does not know the initial location of the evader, then there* *exists an initial pursuer move that allows him to obtain the exact evader location, after which he can continue with S_p.*

Proof. Suppose the pursuer starts at P and the evader at E such that the coordinates of E lie in between the coordinates of P and the origin of the first quadrant. Further, suppose the evader moves to E' and the pursuer has the bearing ray through E'. Call this ray $r(P)$. Our idea is for the pursuer to move to a point P' and then obtain his sensing ray $r(P')$. The intersection of $r(P)$ and $r(P')$ gives the pursuer the exact location of E' and now he continues with S_p.

Note that P' has to be chosen in such a way that Invariant 1 is not broken and the rays $r(P)$ and $r(P')$ do not coincide. There are two cases that arise.

Suppose that E' lies between the pursuer and the origin, then the pursuer can simply move parallel to one of the axes, toward the evader and the new sensing ray will give him the required information. Invariant 1 is clearly preserved.

In the event that E' crosses the pursuer along one of his coordinates, say Y (the other coordinate follows a symmetric argument), the pursuer moves one unit away from the origin parallel to the Y-axis. This guarantees that Invariant 1 is preserved because they initially started with a positive Y-separation. Further, the intersection of the new sensing ray with the previous one gives the pursuer the exact location of the evader and he plays the rest of the game following our strategy S_p. □

4.4.3 An upper bound on guarding distance

In this section, we provide an upper bound on the guarding distance for the guard phase of our bearing-only pursuer strategy (See Section 4.3.2). In the event that the pursuer's conservative move ends up being a wrong guess, he switches to the guarding strategy, explained in Section 4.3.3. We bound the distance that the pursuer is off from the evader's projection as follows.

Lemma 4.4.5. *Let the sensing ray intersect the pursuer's circle before the evader's circle, i.e., the evader lies in the segment E^+E^- (see Figure 4.9), then for $s \geq 1$, $|P^+P^-| \leq 1$ with equality occurring at $s = \sqrt{3}$ and $\alpha = \pi/6$.*

Proof. We first assume that $s \geq \sqrt{3}$ and $\angle E^+EP \geq \pi/2$, i.e., E^+ and E^- lie on the opposite side of the line through E perpendicular to EP. With reference to Figure 4.9, we have

In ΔP^+PU,

$$\begin{aligned} PU &= P^+P\cos\theta = \cos\theta \qquad\qquad (\because P^+P = 1) \\ \Rightarrow EF &= PU = \cos\theta \end{aligned}$$

$\therefore$ From ΔE^+EF

$$\begin{aligned} \angle E^+EF &= \theta \Rightarrow PP^+ \parallel E^+E \\ \Rightarrow \angle E^+EI &= \angle P^+PT \end{aligned}$$

Now $\Delta HEI \equiv \Delta P^-PT$ gives us

$$\begin{aligned} &\angle HEI = \angle P^-PT \\ &\Rightarrow \angle E^+EH = \angle P^+PT - \angle P^-PT = \phi \\ &\Rightarrow \Delta E^+EH \equiv \Delta P^+PP^- \\ &(\because HE = P^-P = E^+E = P^+P = 1) \\ &\Rightarrow \; E^+H = P^+P^- = 2\sin\frac{\phi}{2} \end{aligned}$$

Now

$$\angle E^+EH = \phi = 2\angle E^+E^-H = 2\alpha$$

where the last line comes from the fact that the angle subtended by the arc E^+H at the center of the circle is twice the angle subtended at the circumference. Thus we have $|P^+P^-| = 2\sin\alpha$ which shows that $|P^+P^-|$ increases with α for any fixed s. Moreover, we had assumed that $\angle E^+EP \geq \pi/2$ and therefore $|P^+P^-|$ is maximum when the angle α is such that E^+E is perpendicular to EP. If we increase the angle α further, so that $\angle E^+EP < \pi/2$ we can easily see that $|P^+P^-|$ is less than the case where $\angle E^+EP = \pi/2$. Therefore, for any fixed s, $|P^+P^-|$ is maximum for the value of α for which $\angle E^+EP = \pi/2$ which implies $|P^+P^-|$ is maximum for

$\tan\alpha = 1/s$. So, for $s \geq \sqrt{3}$, α is maximum and equal to $\pi/6$ for $s = \sqrt{3}$ with $|P^+P^-| = 1$ (Figure 4.13 shows the geometry for the optimal solution).

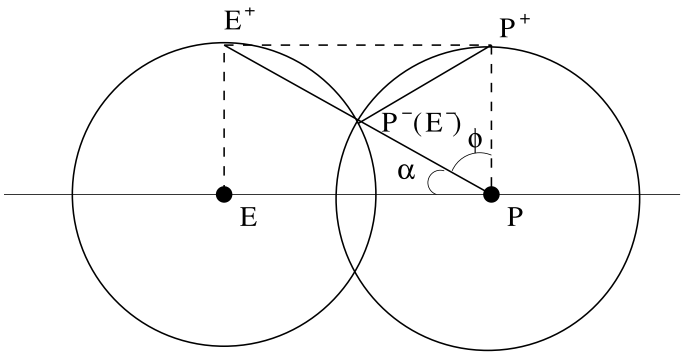


Figure 4.13: The maximum value of $|P^+P^-|$ is 1 and it occurs for $|PE| = \sqrt{3}, \alpha = \pi/6,$ and $\phi = \pi/3$. The line E^+E is perpendicular to EP and the points P^-, E^- are coincident and lie at the intersection of the pursuer's and evader's circle.

Now, if we consider $s < \sqrt{3}$, it is obvious from Figure 4.13 that for all possible α for which $\angle E^+EP \geq \pi/2$, the sensing ray intersects the evader's circle before the pursuer's circle. This case is treated in the next lemma and so we ignore this here. For $\angle E^+EP < \pi/2$, the sensing ray may intersect the evader's circle inside the arc E^+P^- (to see this from Figure 4.13 imagine that the pursuer's circle is pushed to the left and draw a line from P with $\angle E^+EP < \pi/2$ that intersects the evader's circle before the pursuer's circle). Therefore the intercept on the pursuer's circle obtained by drawing lines parallel to EP through $E+$ and E^- lies within the arc P^+P^-. Thus, for $s \leq \sqrt{3}$ the length of the chord P^+P^- is always less than that obtained for $\alpha = \pi/6$ and $s = \sqrt{3}$. This gives our result: for $s \geq 1$, $|P^+P^-| \leq 1$ with equality occuring at $s = \sqrt{3}$ and $\alpha = \pi/6$. □

Lemma 4.4.6. *Let the sensing ray intersect the evader's circle before the pursuer's circle, i.e., the point P^- lies in the line segment E^+E^- (see Figure 4.10), then for $1 \leq s \leq 2$, $|P^+P^-| \leq 1$ with equality occuring at $s = \sqrt{3}$ and $\alpha = \pi/6$.*

Proof. We first assume that $s \leq \sqrt{3}$. With reference to Figure 4.10, we have

$$EF = PU = \cos\theta \Rightarrow \angle E^+EF = \theta$$

$$\Rightarrow HE = E^+F = \sin\theta,\ E^+H = EF = \cos\theta$$

$\therefore$ From ΔE^+PH,

$$\tan\alpha = \frac{E^+H}{PH} = \frac{E^+H}{HE+PE} = \frac{\cos\theta}{\sin\theta + s}$$

$$\Rightarrow s\sin\alpha = \cos(\alpha+\theta)$$

$$\Rightarrow \boxed{s\sin\alpha = \sin\phi} \qquad\qquad (\because \alpha+\theta+\phi = \frac{\pi}{2})$$

Now $s < \sqrt{3}$ and $\phi \geq \frac{\pi}{3}$

$$\Rightarrow \sin\alpha > \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} = \frac{1}{2} \Rightarrow \alpha > \frac{\pi}{6}$$

But $\alpha + \phi \leq \frac{\pi}{2}$ contradiction

$\Rightarrow s < \sqrt{3}$ and $\phi \geq \frac{\pi}{3}$ not possible!

$\Rightarrow P^+P^- \leq 1$ for $s < \sqrt{3}$

If $s = \sqrt{3}, \phi > \frac{\pi}{3} \Rightarrow \sin\alpha > \frac{1}{2} \Rightarrow \alpha > \frac{\pi}{6}$

which leads to a contradiction because $\alpha + \phi \leq \pi/2$. Thus, for $s = \sqrt{3}$, $\phi > \pi/3$ is not possible; the maximum value of ϕ is $\pi/3$ with $\alpha = \pi/6$ and $|P^+P^-| = 2\sin\phi/2 = 1$. From the above we can conclude that $|P^+P^-| \leq 1$ for $s \leq \sqrt{3}$ with equality occuring for $s = \sqrt{3}$.

For $\sqrt{3} < s < 2$, with reference to Figure 4.10

$$E^+H = P^+R > P^+P^-$$

where the equality has already been proved as a part of Lemma 4.4.5. Moreover, in the course of proving Lemma 4.4.5 we have proved that $E^+H \leq 1$ for $s \geq \sqrt{3}$ (to see this note that the point H is formed by the intersection of the line through E^- parallel to EP with the pursuer's cirlce in both Figures 4.9 and 4.10). Thus, we have $|P^+P^-| < 1$ for $\sqrt{3} < s \leq 2$ and it follows that for $1 \leq s \leq 2$, $|P^+P^-| \leq 1$ with equality occuring at $s = \sqrt{3}$ and $\alpha = \pi/6$. $\square$

5. CONCLUSION

In this thesis we studied the effect of reducing the sensing information available to the players of a pursuit-evasion game in the context of two well-known scenarios - the cop and robber game and the lion and man game. Both formulations find applications in search-and-rescue, surveillance, air-traffic control, communication networks and warfare.

In the standard cops and robbers game, we studied the effect of reducing the pursuer's visibility. We showed that a pursuer with limited visibility is as powerful as a pursuer with global visibility in terms of the outcome of the game. On the other hand, we showed that the reduction in visibility can cause an exponential increase in the capture time. We also initiated the study of the cops and robbers game when the players have limited but symmetric visibility powers. For this version, we showed that the cop can establish eye contact with the robber on any graph. Next, we presented a characterization of graphs where a natural greedy strategy suffices for capture. This condition yields a sufficient condition for capture in the symmetric visibility case.

Our results shed light onto the role of information available to the pursuer on the outcome of this game and raises a number of interesting questions for future research. We have shown that a pursuer with limited visibility can still win on a cop-win graph but the capture time may increase exponentially. First, in applications where the capture time is crucial, the decrease in the sensing powers of the pursuers can be compensated by increasing the number of pursuers. What is a sufficient number of pursuers (with limited visibility) to make the capture time polynomial in the number of vertices? Next, on a graph which is not cop-win, what is the number of pursuers (with limited visibility) sufficient to capture the evader? It would be interesting to study how the outcome of the game changes when the players have symmetric visibility. It seems unlikely that the exponential increase in the capture time holds for this case.

In the second part of the thesis, we studied a variant of the well-known lion and man game. We reduced the sensing information of the lion (pursuer) from complete information to bearing-only information. We showed that the lion is able to get to within a step-size distance of the man. Although this outcome is similar to the complete visibility case, the reduction in sensing information costs the lion valuable capture time. Further, when the capture criterion is zero distance, we proved that the man wins with high probability.

It is evident that the reduction in sensing information for the pursuer allows the evader to exploit the uncertainty region in such a way as to change the outcome of the game when the capture distance is zero. The case when the capture threshold is between zero and one unit requires further investigation as it is not yet clear as to what a winning strategy is for either player. We assumed that the pursuer can localize itself precisely. Another direction of future research is to incorporate uncertainties regarding the pursuer's location into the pursuit strategy.

By studying the effect of reduced sensing information in pursuit-evasion tasks, we envision to translate the pursuit strategies and evasion strategies into mobile robot controllers that can achieve the goals of these tasks in practice. As part of an educational outreach program, we have successfully demonstrated how the greedy pursuit strategy can be executed on our mobile robot testbed. The application was for one of our mobile robots to track a red color blob using its camera and follow it. Our implementation threw light on practical issues such as mobile robot control and synchronization, blob detection and image processing, network speed and connectivity, and the human controller interface. In the future, we strive to achieve a synergistic balance between strong theoretical results and system deployment to help mobile robots execute useful tasks in complex environments.

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