TELESEISMIC WAVEFORM MODELING IN 2.5 DIMENSIONS

By

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ABSTRACT

In this thesis we adapt a $2D$ frequency domain finite difference method previously developed for active source seismological wave propagation to now calculate the $2.5D$ elastic response of a local media to teleseismic waves. Since the method is spectral and the discretization is high order it is well equipped to run on moderately sized computers. To prevent us from having to simulate an entire Earth response, we adopt a total field scattered field approach with an analytic solution built on the $1D$ Haskell propagator method. We also adapt a method for reducing numerical artifacts such as anisotropy and dispersion arising from discretization.

INTRODUCTION

Full waveform tomography is a class of methods that attempts to estimate medium properties from the entire seismic waveform. The method is advantageous to linear methods such as Born approximation techniques, in that we are no longer limited by single scattering approximations. When compared to time domain methods, which are bound by the Fresnel zone, we now shift our limiter on information recovery to the dominant wavelength of the signal. As demonstrated by Pratt (2003) this spectral domain method provides access to higher frequency, higher resolution information, with less computational overhead.

To recover a full waveform we must first provide an estimated waveform via forward modeling. Appropriate variational methods such as finite elements (Hughes, 1987), spectral elements (Komatisch and Vilotte, 1998) boundary element methods (Bouchon, 1995 and Fujiwara, 2000) , are all applicable to the inversion method of Pratt et al (1999), and our versatile total field scattered field approach in specifying a plane wave, which is not subject to edge effects as are similar approaches. However, we choose the computationally advantageous and conceptually more direct method of finite differences at the cost of rigid grid specification.

Another computational advantage of the method proposed here is we approximate $3D$ methods with the geologically reasonable assumption of invariance in one direction (Takenaka and Kennett, 1996). This also allows us greater freedom in modeling events that are not necessarily on axis with our regional network.

CHAPTER 1
The Forward Problem

1.1 The Continuous Forms

In this section we begin with the continuous equations of motion for a linearly elastic isotropic hetergenous medium in three dimensions. Since we expect a material invariance parallel to the strike of a convergent event we can apply the simplifying assumption to $2.5D$ dimensions (Takenaka and Kennett, 1996). From there we introduce our artificial absorbing boundaries as described by Zheng and Huang (2002), from now referred to as ZH02. Next we discretize via a 9-point mixed frame finite difference stencil (Stekl and Pratt, 1998). To find the rotated frame will we directly work from our discretized visco-elastic equations. When accumulating our finite difference forms we directly introduce a mass lumping to the intertial term (Jo et. al, 1996). The optimization scheme is further eleaborated upon in the section on numerical dispersion.

1.1.1 Non-Rotated Continuous Equations of Motion

In defining the forward problem we begin with the the general elastic wave equations for an isotropic medium is given by Aki and Richards, (2002):

$$\rho\ddot{U}_i = f_i + \left(\lambda\delta_{ij}e_{kk} + 2\mu e_{ij}\right)_{,j} \tag{1.1}$$

To convert to the frequency domain, we apply the unitary Fourier transform

$$u(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} U(t)e^{i\omega t}dt \tag{1.2}$$

where ω is angular frequency. This form deviates from standard conventions in the lack of the minus sign in the exponent, but is consistent with Pratt's definition of

the Fourier transform. Regardless, we rewrite ((1.1)) as

$$\omega^2 \rho \ddot{u}_i + \left(\lambda \delta_{ij} e_{kk} + 2\mu e_{ij}\right)_{,j} + f_i = 0 \tag{1.3}$$

This expands to

$$\begin{aligned}
&\omega^2 \rho u + \frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z} \\
&\qquad \left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + f_x = 0 \\
&\omega^2 \rho v + \frac{\partial}{\partial y}\left[(\lambda+2\mu)\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z} \\
&\qquad \left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + f_y = 0 \\
&\omega^2 \rho w + \frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y} \\
&\qquad \left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + f_z = 0
\end{aligned}$$

To make the problem computationally palatable we will assume in (1.3) that material variation in y vanish and reduce to 2.5 dimensions, i.e. $\frac{\partial \lambda}{\partial y} = \frac{\partial \mu}{\partial y} = 0$. The other key implication of our assumption is that the wavefield at any position y is a time delayed version from a previous y, i.e.

$$\mathbf{U}(x, y, z, t) = \mathbf{U}(x, 0, z, t - p_y y) \tag{1.4}$$

where p_y is the slowness in the y direction. For a frequency domain representation, we consider a plane wave (with sign convention consistent with our definition of the Fourier Transform in (1.2))

$$\begin{aligned}
\mathbf{u} =& \mathbf{A}(\omega) e^{i(\omega t - k_x x - k_y y - k_z z)} \\
=& \mathbf{A}(\omega) e^{i(\omega(t - p_y y)) - k_x x - k_z z)} \\
=& \mathbf{A}(\omega) e^{i(\omega t - \omega p_y y - k_x x - k_z z)}
\end{aligned}$$

where we recognize a relationship for wavenumber and slowness in y as $k_y = \omega p_y$. It follows that the derivatives in y become

$$\frac{\partial \mathbf{u}}{\partial y} = -i\omega p_y \mathbf{u} \qquad \frac{\partial^2 \mathbf{u}}{\partial y^2} = -(\omega p_y)^2 \mathbf{u}$$

We can now rewrite ((1.3)) as

$$\begin{aligned}
\omega^2\rho u + \frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\left(-i\omega p_y v + \frac{\partial w}{\partial z}\right)\right] + \mu\left(-(\omega p_y)^2 u - i\omega p_y \frac{\partial v}{\partial x}\right) \\
+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + f_x = 0 \\
\omega^2\rho v - (\omega p_y)^2(\lambda+2\mu)v - i\lambda\omega p_y\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} - i\omega p_y u\right)\right] \\
+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} - i\omega p_y w\right)\right] + f_y = 0 \\
\omega^2\rho w + \frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial x} - i\omega p_y v\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \\
+\mu\left(-(\omega p_y)^2 w - i\omega p_y \frac{\partial v}{\partial z}\right) + f_z = 0
\end{aligned}$$

or simplified to

$$\begin{aligned}
(\rho - p_y^2\mu)\omega^2 u + \frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\
- i\omega p_y\left(\frac{\partial}{\partial x}\lambda v + \mu\frac{\partial v}{\partial x}\right) + f_x = 0 \quad (1.5a)
\end{aligned}$$

$$\begin{aligned}
(\rho - p_y^2(\lambda+2\mu))\omega^2 v + \frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) \\
- i\omega p_y\left[\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial x}\mu u + \frac{\partial}{\partial z}\mu w\right] + f_y = 0 \quad (1.5b)
\end{aligned}$$

$$(\rho - p_y^2\mu)\omega^2 w + \frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z} + \lambda\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$
$$- i\omega p_y\left(\frac{\partial}{\partial z}\lambda v + \mu\frac{\partial v}{\partial z}\right) + f_z = 0 \quad (1.5c)$$

Observe that though we still have to solve for (u, v, w), the differencing matrix is now $3 \times n_x \times n_z$ instead of $n_x \times n_y \times n_z$.

1.1.2 Non Rotated Equations of Motions in the PML

Since we can not computationally propagate a wave in an infinite layer, we must create a faux infinite layer via the Perfectly Matched Layer (PML). Following ZH02 we can create a complex stretching in the absorbing boundary which will attenuate incoming waves even at grazing incidence. For this reason, we prefer this method to a simpler technique such as the Clayton Engquist absorbing boundary condition (Clayton and Engquist, 1977). An additional advantage of ZH02 formulation of is that it readily available for implementation in a non-cartesian frame, an advantage over the formulation by Komatisch and Tromp (2003). Starting with equation (21) of ZH02 we have a stretched coordinate form of (1.1)

$$-\rho\omega^2\epsilon_x\epsilon_y\epsilon_z u_i = \hat{\sigma}_{ij,j} \tag{1.6}$$

Upon substituting (22) - (30) of ZH02 we write out the stress tensor in (1.6) as

$$\frac{\partial\hat{\sigma}_{xx}}{\partial x} = \frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\epsilon_y\epsilon_z}{\epsilon_x}\frac{\partial u}{\partial x} + \lambda\epsilon_z\frac{\partial v}{\partial y} + \lambda\epsilon_y\frac{\partial w}{\partial z}\right] \tag{1.7a}$$

$$\frac{\partial\hat{\sigma}_{yx}}{\partial y} = \frac{\partial}{\partial y}\left[\mu\epsilon_z\frac{\partial v}{\partial x} + \mu\frac{\epsilon_x\epsilon_z}{\epsilon_y}\frac{\partial u}{\partial y}\right] \tag{1.7b}$$

$$\frac{\partial\hat{\sigma}_{zx}}{\partial z} = \frac{\partial}{\partial z}\left[\mu\epsilon_y\frac{\partial w}{\partial x} + \mu\frac{\epsilon_x\epsilon_y}{\epsilon_z}\frac{\partial u}{\partial z}\right] \tag{1.7c}$$

$$\frac{\partial\hat{\sigma}_{yy}}{\partial y} = \frac{\partial}{\partial y}\left[(\lambda+2\mu)\frac{\epsilon_x\epsilon_z}{\epsilon_y}\frac{\partial v}{\partial y} + \lambda\epsilon_z\frac{\partial u}{\partial x} + \lambda\epsilon_x\frac{\partial w}{\partial z}\right] \tag{1.7d}$$

$$\frac{\partial\hat{\sigma}_{xy}}{\partial x} = \frac{\partial}{\partial x}\left[\mu\epsilon_z\frac{\partial u}{\partial y} + \mu\frac{\epsilon_y\epsilon_z}{\epsilon_x}\frac{\partial v}{\partial x}\right] \tag{1.7e}$$

$$\frac{\partial\hat{\sigma}_{zy}}{\partial z} = \frac{\partial}{\partial z}\left[\mu\epsilon_x\frac{\partial w}{\partial y} + \mu\frac{\epsilon_x\epsilon_y}{\epsilon_z}\frac{\partial v}{\partial z}\right] \tag{1.7f}$$

$$\frac{\partial\hat{\sigma}_{zz}}{\partial z} = \frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\epsilon_x\epsilon_y}{\epsilon_z}\frac{\partial w}{\partial z} + \lambda\epsilon_y\frac{\partial u}{\partial x} + \lambda\epsilon_x\frac{\partial v}{\partial y}\right] \tag{1.7g}$$

$$\frac{\partial\hat{\sigma}_{xz}}{\partial x} = \frac{\partial}{\partial x}\left[\mu\epsilon_y\frac{\partial u}{\partial z} + \mu\frac{\epsilon_y\epsilon_z}{\epsilon_x}\frac{\partial w}{\partial x}\right] \tag{1.7h}$$

$$\frac{\partial\hat{\sigma}_{yz}}{\partial y} = \frac{\partial}{\partial y}\left[\mu\epsilon_x\frac{\partial v}{\partial z} + \mu\frac{\epsilon_x\epsilon_z}{\epsilon_y}\frac{\partial w}{\partial y}\right] \tag{1.7i}$$

For u we find

$$-\omega^2\epsilon_x\epsilon_y\epsilon_z\rho u = \frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\epsilon_y\epsilon_z}{\epsilon_x}\frac{\partial u}{\partial x} + \lambda\epsilon_z\frac{\partial v}{\partial y} + \lambda\epsilon_y\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial y}\left[\mu\epsilon_z\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\frac{\epsilon_x\epsilon_z}{\epsilon_y}\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\epsilon_y\frac{\partial w}{\partial x} + \mu\frac{\epsilon_x\epsilon_y}{\epsilon_z}\frac{\partial u}{\partial z}\right]$$

or

$$-\omega^2\epsilon_x\epsilon_y\epsilon_z\rho u = \epsilon_y\epsilon_z\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{1}{\epsilon_x}\frac{\partial u}{\partial x}\right] + \epsilon_z\frac{\partial}{\partial x}\left[\lambda\frac{\partial v}{\partial y}\right] + \epsilon_y\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right] + \epsilon_z\frac{\partial}{\partial y}\left[\mu\frac{\partial v}{\partial x}\right] + \epsilon_x\epsilon_z\frac{\partial}{\partial y}\left[\mu\frac{1}{\epsilon_y}\frac{\partial u}{\partial y}\right] + \epsilon_y\frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right] + \epsilon_x\epsilon_y\frac{\partial}{\partial z}\left[\mu\frac{1}{\epsilon_z}\frac{\partial u}{\partial z}\right]$$

Then in v

$$-\omega^2\epsilon_x\epsilon_y\epsilon_z\rho v = \frac{\partial}{\partial y}\left[(\lambda+2\mu)\frac{\epsilon_x\epsilon_z}{\epsilon_y}\frac{\partial v}{\partial y} + \lambda\epsilon_z\frac{\partial u}{\partial x} + \lambda\epsilon_x\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial x}\left[\mu\epsilon_z\frac{\partial u}{\partial y} + \mu\frac{\epsilon_y\epsilon_z}{\epsilon_x}\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial z}\left[\mu\epsilon_x\frac{\partial w}{\partial y} + \mu\frac{\epsilon_x\epsilon_y}{\epsilon_z}\frac{\partial v}{\partial z}\right]$$

or

$$-\omega^2\epsilon_x\epsilon_y\epsilon_z\rho v = \epsilon_x\epsilon_z\frac{\partial}{\partial y}\left[(\lambda+2\mu)\frac{1}{\epsilon_y}\frac{\partial v}{\partial y}\right] + \epsilon_z\frac{\partial}{\partial y}\left[\lambda\frac{\partial u}{\partial x}\right] + \epsilon_x\frac{\partial}{\partial y}\left[\lambda\frac{\partial w}{\partial z}\right] + \epsilon_z\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial y}\right] + \epsilon_y\epsilon_z\frac{\partial}{\partial x}\left[\mu\frac{1}{\epsilon_x}\frac{\partial v}{\partial x}\right] + \epsilon_x\frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial y}\right] + \epsilon_x\epsilon_y\frac{\partial}{\partial z}\left[\mu\frac{1}{\epsilon_z}\frac{\partial v}{\partial z}\right]$$

And for w

$$-\omega^2\epsilon_x\epsilon_y\epsilon_z\rho w = \frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\epsilon_x\epsilon_y}{\epsilon_z}\frac{\partial w}{\partial z} + \lambda\epsilon_y\frac{\partial u}{\partial x} + \lambda\epsilon_x\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial x}\left[\mu\epsilon_y\frac{\partial u}{\partial z} + \mu\frac{\epsilon_y\epsilon_z}{\epsilon_x}\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\epsilon_x\frac{\partial v}{\partial z} + \mu\frac{\epsilon_x\epsilon_z}{\epsilon_y}\frac{\partial w}{\partial y}\right]$$

or

$$-\omega^2\epsilon_x\epsilon_y\epsilon_z\rho w = \epsilon_x\epsilon_y\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{1}{\epsilon_z}\frac{\partial w}{\partial z}\right] + \epsilon_y\frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right] + \epsilon_x\frac{\partial}{\partial z}\left[\lambda\frac{\partial v}{\partial y}\right] + \epsilon_y\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right] + \epsilon_y\epsilon_z\left[\mu\frac{1}{\epsilon_x}\frac{\partial w}{\partial x}\right] + \epsilon_x\frac{\partial}{\partial y}\left[\mu\frac{\partial v}{\partial z}\right] + \epsilon_x\epsilon_z\frac{\partial}{\partial y}\left[\mu\frac{1}{\epsilon_y}\frac{\partial w}{\partial y}\right]$$

For 2.5D media we note $\epsilon_y = 1$ and $\frac{\partial}{\partial y} = -i\omega p_y$. This allows us to simplify the above for u to

$$-\omega^2\epsilon_x\epsilon_z\rho u = \epsilon_z\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{1}{\epsilon_x}\frac{\partial u}{\partial x}\right] - i\omega p_y\epsilon_z\frac{\partial}{\partial x}\left[\lambda v\right] + \frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right] - i\omega p_y\epsilon_z\mu\frac{\partial v}{\partial x} - (\omega p_y)^2\epsilon_x\epsilon_z\mu u + \frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right] + \epsilon_x\frac{\partial}{\partial z}\left[\mu\frac{1}{\epsilon_z}\frac{\partial u}{\partial z}\right]$$

Then, dividing through by the ϵ's on the left and cleaning up some we find in u

$$(\mu p_y^2 - \rho)\omega^2 u = \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{1}{\epsilon_x}\frac{\partial u}{\partial x}\right] + \frac{1}{\epsilon_x\epsilon_z}\left(\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]\right) + \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[\mu\frac{1}{\epsilon_z}\frac{\partial u}{\partial z}\right] - i\frac{\omega p_y}{\epsilon_x}\left[\frac{\partial}{\partial x}[\lambda v] + \mu\frac{\partial v}{\partial x}\right] \quad (1.8)$$

similarly in v

$$-\omega^2\epsilon_x\epsilon_z\rho v = -(\omega p_y)^2\epsilon_x\epsilon_z(\lambda+2\mu)v - i\omega p_y\left[\epsilon_z\lambda\frac{\partial u}{\partial x} + \epsilon_x\lambda\frac{\partial w}{\partial z}\right] - i\omega p_y\epsilon_z\frac{\partial}{\partial x}[\mu u] + \epsilon_z\frac{\partial}{\partial x}\left[\mu\frac{1}{\epsilon_x}\frac{\partial v}{\partial x}\right] - i\omega p_y\epsilon_x\frac{\partial}{\partial z}\left[\mu w\right] + \epsilon_x\frac{\partial}{\partial z}\left[\mu\frac{1}{\epsilon_z}\frac{\partial v}{\partial z}\right]$$

or

$$(p_y^2(\lambda+2\mu) - \rho)\omega^2 v = \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[\mu\frac{1}{\epsilon_x}\frac{\partial v}{\partial x}\right] + \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[\mu\frac{1}{\epsilon_z}\frac{\partial v}{\partial z}\right] - i\omega p_y\left(\frac{1}{\epsilon_x}\left\{\frac{\partial}{\partial x}[\mu u] + \lambda\frac{\partial u}{\partial x}\right\} + \frac{1}{\epsilon_z}\left\{\frac{\partial}{\partial z}[\mu w] + \lambda\frac{\partial w}{\partial z}\right\}\right) \quad (1.9)$$

and likewise in w

$$-\omega^2\epsilon_x\epsilon_z\rho w = \epsilon_x\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{1}{\epsilon_z}\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right] - i\omega p_y\epsilon_x\frac{\partial}{\partial z}[\lambda v] + \frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right] + \epsilon_z\left[\mu\frac{1}{\epsilon_x}\frac{\partial w}{\partial x}\right] - i\omega p_y\epsilon_x\mu\frac{\partial v}{\partial z} - (\omega p_y)^2\epsilon_x\epsilon_z\mu w$$

or

$$(p_y^2\mu-\rho)\omega^2 w = \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{1}{\epsilon_z}\frac{\partial w}{\partial z}\right] + \frac{1}{\epsilon_x\epsilon_z}\left(\frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right]\right) + \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[\mu\frac{1}{\epsilon_x}\frac{\partial w}{\partial x}\right] - i\frac{\omega p_y}{\epsilon_z}\left(\mu\frac{\partial v}{\partial z} + \frac{\partial}{\partial z}[\lambda v]\right) \quad (1.10)$$

Then expanding the derivatives on (1.8), (1.9), and (1.10) we get

$$(\mu p_y^2-\rho)\omega^2 u = \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[\frac{1}{\epsilon_x}\right]\left((\lambda+2\mu)\frac{\partial u}{\partial x}\right) + \frac{1}{\epsilon_x^2}\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x}\right] + \frac{1}{\epsilon_x\epsilon_z}\left(\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]\right) + \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[\frac{1}{\epsilon_z}\right]\mu\frac{\partial u}{\partial z} + \frac{1}{\epsilon_z^2}\frac{\partial}{\partial z}\left[\mu\frac{\partial u}{\partial z}\right] - i\frac{\omega p_y}{\epsilon_x}\left(\frac{\partial}{\partial x}[\lambda v] + \mu\frac{\partial v}{\partial x}\right) \quad (1.11a)$$

$$(p_y^2(\lambda+2\mu)-\rho)\omega^2 v = \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[\frac{1}{\epsilon_x}\right]\mu\frac{\partial v}{\partial x} + \frac{1}{\epsilon_x^2}\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right] + \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[\frac{1}{\epsilon_z}\right]\mu\frac{\partial v}{\partial z} + \frac{1}{\epsilon_z^2}\frac{\partial}{\partial z}\left[\mu\frac{\partial v}{\partial z}\right] - i\omega p_y\left(\frac{1}{\epsilon_x}\left\{\frac{\partial}{\partial x}[\mu u] + \lambda\frac{\partial u}{\partial x}\right\} + \frac{1}{\epsilon_z}\left\{\frac{\partial}{\partial z}[\mu w] + \lambda\frac{\partial w}{\partial z}\right\}\right) \quad (1.11b)$$

$$(\mu p_y^2-\rho)\omega^2 w = \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[\frac{1}{\epsilon_z}\right]\left((\lambda+2\mu)\frac{\partial w}{\partial z}\right) + \frac{1}{\epsilon_z^2}\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z}\right] + \frac{1}{\epsilon_x\epsilon_z}\left(\frac{\partial}{\partial x}\left[\lambda\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right]\right) + \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[\frac{1}{\epsilon_x}\right]\mu\frac{\partial w}{\partial x} + \frac{1}{\epsilon_x^2}\frac{\partial}{\partial x}\left[\mu\frac{\partial w}{\partial x}\right] - i\frac{\omega p_y}{\epsilon_z}\left(\mu\frac{\partial v}{\partial z} + \frac{\partial}{\partial z}[\lambda v]\right) \quad (1.11c)$$

From here we can define

$$R_{1x} \equiv \frac{1}{\epsilon_x} \qquad R_{2x} \equiv \frac{1}{\epsilon_x}\frac{\partial}{\partial x}\left[\frac{1}{\epsilon_x}\right] \tag{1.12a}$$

$$R_{1z} \equiv \frac{1}{\epsilon_z} \qquad R_{2z} \equiv \frac{1}{\epsilon_z}\frac{\partial}{\partial z}\left[\frac{1}{\epsilon_z}\right] \tag{1.12b}$$

We can now rewrite (1.11a), (1.11b), and (1.11c) as

$$\begin{aligned}(\mu p_y^2 - \rho)\omega^2 u = {} & R_{2x}(\lambda+2\mu)\frac{\partial u}{\partial x} + R_{1x}^2\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x}\right] + R_{1x}R_{1z}\left(\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]\right) \\ & + R_{2z}\mu\frac{\partial u}{\partial z} + R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial u}{\partial z}\right] - R_{1x}i\omega p_y\left(\frac{\partial}{\partial x}\left[\lambda v\right] + \mu\frac{\partial v}{\partial x}\right)\end{aligned} \tag{1.13a}$$

$$\begin{aligned}(p_y^2(\lambda+2\mu) - \rho)\omega^2 v = {} & R_{2x}\mu\frac{\partial v}{\partial x} + R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right] + R_{2z}\mu\frac{\partial v}{\partial z} + R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial v}{\partial z}\right] \\ & - i\omega p_y\left(R_{1x}\left\{\frac{\partial}{\partial x}\left[\mu u\right] + \lambda\frac{\partial u}{\partial x}\right\} + R_{1z}\left\{\frac{\partial}{\partial z}\left[\mu w\right] + \lambda\frac{\partial w}{\partial z}\right\}\right)\end{aligned} \tag{1.13b}$$

$$\begin{aligned}(\mu p_y^2 - \rho)\omega^2 w = {} & R_{2z}(\lambda+2\mu)\frac{\partial w}{\partial z} + R_{1z}^2\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z}\right] + R_{1x}R_{1z}\left(\frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right]\right) \\ & + R_{2x}\mu\frac{\partial w}{\partial x} + R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial w}{\partial x}\right] - R_{1z}i\omega p_y\left(\frac{\partial}{\partial z}\left[\lambda v\right] + \mu\frac{\partial v}{\partial z}\right)\end{aligned} \tag{1.13c}$$

1.1.3 Rotated Continuous Equations of Motion in the PML

For greater numerical accuracy we seek a rotated discretization (Štekl and Pratt, 1998). To introduce the rotated coordinates we take x to be positive right, z to be positive down. We include a 45° clockwise rotated system by letting (x', z') be the displacements related to (x, z) by

$$\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

Under a clockwise $(-45°)$ rotation we find

$$x' = \tfrac{x+z}{\sqrt{2}}; \quad z' = \tfrac{z-x}{\sqrt{2}}$$

and back with $(+45°)$

$$x = \tfrac{x'-z'}{\sqrt{2}}; \quad z = \tfrac{x'+z'}{\sqrt{2}}$$

Taking the first derivatives we find

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial x} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial z'}\right) \tag{1.14a}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial z} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial z'}\right) \tag{1.14b}$$

Then applying (1.14a), (1.14b) to (1.13a) (1.13b) and (1.13c) we find

$$\begin{aligned}(\mu p_y^2 - \rho)\omega^2 u = {} & \frac{R_{2x}}{\sqrt{2}}(\lambda+2\mu)\left(\frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'}\right) + \frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial z'}\right)\left[(\lambda+2\mu)\left(\frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'}\right)\right] \\ & + \frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial z'}\right)\left[\lambda\left(\frac{\partial w}{\partial x'} + \frac{\partial w}{\partial z'}\right)\right] + \left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial w}{\partial x'} - \frac{\partial w}{\partial z'}\right)\right]\right\} \\ & + \frac{R_{2z}}{\sqrt{2}}\left\{\mu\left(\frac{\partial u}{\partial z'} + \frac{\partial u}{\partial x'}\right)\right\} + \frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'} + \frac{\partial}{\partial x'}\right)\left[\mu\frac{\partial u}{\partial z'} + \frac{\partial u}{\partial x'}\right] \\ & - R_{1x} i\frac{\omega p_y}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\lambda v] - \frac{\partial}{\partial z'}[\lambda v] + \mu\left(\frac{\partial v}{\partial x'} - \frac{\partial v}{\partial z'}\right)\right\}\end{aligned} \tag{1.15a}$$

$$(p_y^2(\lambda+2\mu)-\rho)\omega^2 v = \frac{R_{2x}}{\sqrt{2}}\mu\left(\frac{\partial v}{\partial x'}-\frac{\partial v}{\partial z'}\right)+\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial v}{\partial x'}-\frac{\partial v}{\partial z'}\right)\right]$$
$$+\frac{R_{2z}}{\sqrt{2}}\mu\left(\frac{\partial v}{\partial z'}+\frac{\partial v}{\partial x'}\right)+\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\mu\left(\frac{\partial v}{\partial z'}+\frac{\partial v}{\partial x'}\right)\right]$$
$$-i\omega p_y\frac{R_{1x}}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\mu u]-\frac{\partial}{\partial z'}[\mu u]+\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right\}$$
$$-i\omega p_y\frac{R_{1z}}{\sqrt{2}}\left\{\frac{\partial}{\partial z'}[\mu w]+\frac{\partial}{\partial x'}[\mu w]+\lambda\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right\}$$

$$(p_y^2(\lambda+2\mu)-\rho)\omega^2 v = \frac{\mu}{\sqrt{2}}\left(\frac{\partial v}{\partial x'}(R_{2x}+R_{2z})+\frac{\partial v}{\partial z'}(R_{2z}-R_{2x})\right)$$
$$+\frac{1}{2}\left[(R_{1x}^2+R_{1z}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial x'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial z'}\right]\right)+(R_{1z}^2-R_{1x}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial x'}\right]\right)\right]$$
$$-i\omega p_y\frac{R_{1x}}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\mu u]-\frac{\partial}{\partial z'}[\mu u]+\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right\}$$
$$-i\omega p_y\frac{R_{1z}}{\sqrt{2}}\left\{\frac{\partial}{\partial z'}[\mu w]+\frac{\partial}{\partial x'}[\mu w]+\lambda\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right\} \quad (1.15b)$$

$$(\mu p_y^2-\rho)\omega^2 w = \frac{R_{2z}}{\sqrt{2}}(\lambda+2\mu)\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)+\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[(\lambda+2\mu)\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right]$$
$$+\frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right]+\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right]\right\}$$
$$+\frac{R_{2x}}{\sqrt{2}}\left\{\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right\}+\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right]$$
$$-R_{1z}i\frac{\omega p_y}{\sqrt{2}}\left\{\mu\left(\frac{\partial v}{\partial z'}+\frac{\partial v}{\partial x'}\right)+\frac{\partial}{\partial z'}[\lambda v]+\frac{\partial}{\partial x'}[\lambda v]\right\} \quad (1.15c)$$

1.1.4 A Note on the Pefectly Matched Layer

The stretched coordinate system, $\hat{x}$, in the PML region is defined by

$$\hat{x}_i = x_i^0 + \int_{x_i^0}^{x_i} \epsilon_i(x')\, dx' \tag{1.16}$$

This is equivalent to the Komatisch and Tromp, (2003) definition

$$\hat{x}_i = x_i - \frac{i}{\omega} \int_0^{x_i} d_i(x')\, dx'$$

To show how this works, we set

$$\begin{aligned} x_i^0 + \int_{x_i^0}^{x_i} \epsilon_i(x')\, dx' = &\quad x_i - \tfrac{i}{\omega} \textstyle\int_0^{x_i} d_i(x') dx' \\ x_i^0 + E_i(x_i) - E_i(x_i^0) = &\quad x_i - \tfrac{i}{\omega} \left(D_i(x_i) - D_i(x_i^0)\right) \end{aligned}$$

or

$$E_i(x_i) = x_i - \frac{i}{\omega} D_i(x_i)$$

Finally,

$$\epsilon_i(x) = 1 - \frac{i}{\omega} d_i(x)$$

Also, from the chain rule:

$$\frac{\partial}{\partial \hat{x}_i} = \frac{\partial x_i}{\partial \hat{x}_i} \frac{\partial}{\partial x_i}$$

$$\frac{\partial \hat{x}_i}{\partial x_i} = \frac{\partial}{\partial x_i} \left(x_i^0 + \int_{x_i^0}^{x_i} \epsilon_i(x')\, dx' \right) = \frac{\partial}{\partial x_i} E_i(x_i) = \epsilon_i(x_i)$$

$$\frac{\partial}{\partial \hat{x}_i} = \frac{1}{\epsilon_i} \frac{\partial}{\partial x_i}$$

Note that

$$\frac{\partial x_i}{\partial \hat{x}_i} + = \frac{1}{\epsilon_i(x_i)} = \frac{1}{1 - \frac{i}{\omega} d_i(x_i)} = \frac{\omega}{\omega - i d_i(x_i)} = \frac{i\omega}{i\omega + d_i(x_i)}$$

which is equation 8 of Komatisch and Tromp.

$$\begin{aligned} R_{2x} &= \frac{1}{\epsilon_x}\left(\frac{\partial}{\partial x}\frac{1}{\epsilon_x}\right) \\ &= \frac{1}{\epsilon_x^3}\frac{\partial \epsilon_x}{\partial x} \\ &= -\left(\frac{i\omega}{i\omega + d(x)}\right)^3 \frac{\partial}{\partial x}\left(\frac{i\omega + d(x)}{i\omega}\right) \\ &= -\left(\frac{i\omega}{i\omega + d(x)}\right)^3 \frac{1}{i\omega}\frac{\partial d(x)}{\partial x} \end{aligned}$$

The D Function

$$d(n) = \frac{3\alpha}{2\delta}\log\left(\frac{1}{R}\right)\left(\frac{n}{\delta}\right)^2 = n^2\frac{3\alpha}{2\delta^3}\log\left(\frac{1}{R}\right)$$

$$d'(n) = n\frac{3\alpha}{\delta^3}\log\left(\frac{1}{R}\right)$$

where α is the P wavespeed, δ is the width of the PML layer, n is the distance into the PML, and R is the reflection coefficient chosen as 0.001.

$$\frac{i\omega}{i\omega + d} = \frac{i}{i}\left(\frac{i\omega}{i\omega + d}\right) = \frac{\omega}{\omega - id} = \frac{\omega}{\sqrt{\omega^2 + d^2}e^{-id/\omega}} = \frac{\omega}{\sqrt{\omega^2 + d^2}}e^{id/\omega}$$

d goes like n^2, starting at 0. Note, the two competing items are the reflection coefficient, and width of the layer. Ideally, we would like to choose R extremely small, but to counter act this we must then take δ to be larger thereby keeping d approximately constant. This is because of the introduction of numerical reflection since the computer does not see a smooth exponential decay, but rather a staircase representing the exponent.

A note on signs. The integral

$$\hat{x}_i = x_i^0 + \int_{x_i^0}^{x_i} \epsilon_i(x')\,dx'$$

presumes that we are going in the $+x$ direction. If we are going in the $-x$ direction, we reverse the sign of integration so in effect taking the negative of ϵ. That's okay

but we need to be mindful of the sign on the derivative term

$$\frac{\partial}{\partial x}\left(\frac{1}{\epsilon_x}\right)$$

because x is going in the opposite direction.

$$\epsilon_i(x) = \frac{\omega - id_i(x)}{\omega} = \frac{i\omega + d_i(x)}{i\omega}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\epsilon_x}\right) = \frac{1}{\epsilon_x^2}\frac{\partial \epsilon_x}{\partial x} = -\left(\frac{i\omega}{1\omega + d_i(x)}\right)^2 \frac{\partial}{\partial x}\left(\frac{i\omega + d_i(x)}{i\omega}\right) = -\frac{i\omega}{(i\omega + d_i(x))^2}\frac{\partial d_i(x)}{\partial x}$$

The idea is that we evaluate d with x increasing away from the boundary, but for the $(-x, -z)$ sides this is in the opposite direction, so the derivative term will be negative. This would account for the extra minus sign on this term from Komatisch and Tromp.

1.2 Discretization for the Continuous Equations

1.2.1 The Finite Difference Forms

To solve equations the non-rotated and rotated continuous equations given by 1.13a, 1.13b, 1.13c and 1.15a, 1.15b, 1.15c, we choose the method of finite differences. To introduce the finite difference forms we will use the following stencil:

$$\begin{matrix} f_{i-1,k-1} & f_{i,k-1} & f_{i+1,k-1} \\ f_{i-1,k} & f_{i,k} & f_{i+1,k} \\ f_{i-1,k+1} & f_{i,k+1} & f_{i+1,k+1} \end{matrix} \tag{1.17}$$

1.2.2 Finite Difference Templates

Without loss of generality I will provide differencing templates for a function f which could be u, v, or w, as well as γ which could be any combination of λ and μ.

1.2.3 The Non-Rotated Templates

$$F1 \equiv \frac{\partial}{\partial x}\left[\gamma\frac{\partial f}{\partial x}\right] = \frac{1}{\Delta}\left[(\gamma_{i+1,k}-\gamma_{i,k})\frac{(f_{i+1,k}-f_{i,k})}{\Delta} - (\gamma_{i,k}-\gamma_{i-1,k})\frac{(f_{i,k}-f_{i-1,k})}{\Delta}\right]$$
$$= \frac{1}{\Delta^2}\left[\gamma_{i+1/2\,,k}\,(f_{i+1,k}-f_{i,k}) - \gamma_{i-1/2\,,k}\,(f_{i,k}-f_{i-1,k})\right] \tag{1.18a}$$

$$F2 \equiv \frac{\partial}{\partial x}\left[\gamma\frac{\partial f}{\partial z}\right] = \frac{1}{4\Delta^2}\left[\gamma_{i+1,k}\,(f_{i+1,k+1}-f_{i+1,k-1}) - \gamma_{i-1,k}\,(f_{i-1,k+1}-f_{i-1,k-1})\right] \tag{1.18b}$$

$$F3 \equiv \frac{\partial}{\partial z}\left[\gamma\frac{\partial f}{\partial z}\right] = \frac{1}{\Delta^2}\left[\gamma_{i,k+1/2}\,(f_{i,k+1}-f_{i,k}) - \gamma_{i,k-1/2}\,(f_{i,k}-f_{i,k-1})\right] \tag{1.18c}$$

$$F4 \equiv \frac{\partial}{\partial z}\left[\gamma\frac{\partial f}{\partial x}\right] = \frac{1}{2\Delta}\left[\gamma_{i,k+1}\,\frac{(f_{i+1,k+1}-f_{i-1,k+1})}{2\Delta} - \gamma_{i,k-1}\,\frac{(f_{i+1,k-1}-f_{i-1,k-1})}{2\Delta}\right]$$
$$= \frac{1}{4\Delta^2}\left[\gamma_{i,k+1}\,(f_{i+1,k+1}-f_{i-1,k+1}) - \gamma_{i,k-1}\,(f_{i+1,k-1}-f_{i-1,k-1})\right] \tag{1.18d}$$

$$F5 \equiv \frac{\partial}{\partial x}[\gamma f] = \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\,(f_{i+1,k}+f_{i,k}) - \gamma_{i-1/2\,,k}\,(f_{i,k}+f_{i-1,k})\right] \tag{1.18e}$$

$$F6 \equiv \gamma \frac{\partial f}{\partial x} = \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\left(f_{i+1,k} - f_{i,k}\right) + \gamma_{i-1/2\,,k}\left(f_{i,k} - f_{i-1,k}\right)\right] \tag{1.18f}$$

$$F7 \equiv \gamma \frac{\partial f}{\partial z} = \frac{1}{2\Delta}\left[\gamma_{i,k+1/2}\left(f_{i,k+1} - f_{i,k}\right) + \gamma_{i,k-1/2}\left(f_{i,k} - f_{i,k-1}\right)\right] \tag{1.18g}$$

$$F8 \equiv \frac{\partial}{\partial z}[\gamma f] = \frac{1}{2\Delta}\left[\gamma_{i,k+1/2}(f_{i,k+1} + f_{i,k}) - \gamma_{i,k-1/2}(f_{i,k} + f_{i,k-1})\right] \tag{1.18h}$$

1.2.4 The Rotated Templates

$$RF1 \equiv \gamma \frac{\partial f}{\partial x'} = \frac{1}{2\sqrt{2}\Delta}\left[\gamma_{i+1/2\,,k+1/2}\left(f_{i+1,k+1} - f_{i,k}\right) + \gamma_{i-1/2\,,k-1/2}\left(f_{i,k} - f_{i-1,k-1}\right)\right] \tag{1.19a}$$

$$RF2 \equiv \gamma \frac{\partial f}{\partial z'} = \frac{1}{2\sqrt{2}\Delta}\left[\gamma_{i-1/2\,,k+1/2}\left(f_{i-1,k+1} - f_{i,k}\right) + \gamma_{i+1/2\,,k-1/2}\left(f_{i,k} - f_{i+1,k-1}\right)\right] \tag{1.19b}$$

$$RF3 \equiv \frac{\partial}{\partial x'}\left[\gamma \frac{\partial f}{\partial x'}\right] = \frac{1}{2\Delta^2}\left[\gamma_{i+1/2\,,k+1/2}\left(f_{i+1,k+1} - f_{i,k}\right) - \gamma_{i-1/2\,,k-1/2}\left(f_{i,k} - f_{i-1,k-1}\right)\right] \tag{1.19c}$$

$$RF4 \equiv \frac{\partial}{\partial x'}\left[\gamma \frac{\partial f}{\partial z'}\right] = \frac{1}{2\Delta^2}\left[\gamma_{i+1/2\,,k+1/2}\left(f_{i,k+1} - f_{i+1,k}\right) - \gamma_{i-1/2\,,k-1/2}\left(f_{i-1,k} - f_{i,k-1}\right)\right] \tag{1.19d}$$

$$RF5 \equiv \frac{\partial}{\partial z'}\left[\gamma \frac{\partial f}{\partial x'}\right] = \frac{1}{2\Delta^2}\left[\gamma_{i-1/2\,,k+1/2}\left(f_{i,k+1} - f_{i-1,k}\right) - \gamma_{i+1/2\,,k-1/2}\left(f_{i+1,k} - f_{i,k-1}\right)\right] \tag{1.19e}$$

$$RF6 \equiv \frac{\partial}{\partial z'}\left[\gamma \frac{\partial f}{\partial z'}\right] = \frac{1}{2\Delta^2}\left[\gamma_{i-1/2\,,k+1/2}\left(f_{i-1,k+1} - f_{i,k}\right) - \gamma_{i+1/2\,,k-1/2}\left(f_{i,k} - f_{i+1,k-1}\right)\right] \tag{1.19f}$$

$$RF7 \equiv \frac{\partial}{\partial x'}[\gamma f] = \frac{1}{2\sqrt{2}\Delta}\left[\gamma_{i+1/2\,,k+1/2}\left(f_{i+1,k+1} + f_{i,k}\right) - \gamma_{i-1/2\,,k-1/2}\left(f_{i,k} + f_{i-1,k-1}\right)\right] \tag{1.19g}$$

$$RF8 \equiv \frac{\partial}{\partial z'}[\gamma f] = \frac{1}{2\sqrt{2}\Delta}\left[\gamma_{i-1/2\,,k+1/2}\left(f_{i-1,k+1} + f_{i,k}\right) - \gamma_{i+1/2\,,k-1/2}\left(f_{i,k} + f_{i+1,k-1}\right)\right] \tag{1.19h}$$

1.2.5 Notes on Some Derivatives

In developing the above forms, we have on occasion chosen an intuitive approach to the discretization of a derivative as opposed to a strictest form derived directly from Taylor's expansion. Below, are details of the instances we have chosen

averaging to numerical analysis.

$$\gamma\frac{\partial f}{\partial x} = \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\left(f_{i+1,k} - f_{i,k}\right) + \gamma_{i-1/2\,,k}\left(f_{i,k} - f_{i-1,k}\right)\right]$$

$$\gamma\frac{\partial f}{\partial x} = \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\, f_{i+1,k} - \gamma_{i-1/2\,,k}\, f_{i-1,k} - f_{i,k}\left(\gamma_{i+1/2\,,k} - \gamma_{i-1/2\,k}\right]\right.$$

$$= \frac{1}{4\Delta}\left[\gamma_{i+1,k}\, f_{i+1,k} - \gamma_{i-1,k}\, f_{i-1,k} + \gamma_{i,k}\left(f_{i+1,k} - f_{i-1,k}\right) - f_{i,k}\left(\gamma_{i+1,k} - \gamma_{i-1,k}\right)\right]$$

$$= \frac{1}{4\Delta}\left[\gamma_{i+1,k}\left[f_{i+1,k} - f_{i,k}\right] + \gamma_{i-1,k}\left[f_{i,k} - f_{i-1,k}\right] + \gamma_{i,k}\left[f_{i+1,k} - f_{i-1,k}\right]\right]$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\gamma_{i+1,k}\,\frac{(f_{i+1,k} - f_{i,k})}{\Delta} + \gamma_{i-1,k}\,\frac{(f_{i,k} - f_{i-1,k})}{\Delta}\right] + \gamma_{i,k}\frac{(f_{i+1,k} - f_{i-1,k})}{2\Delta}\right\}$$

Note we could compute $\frac{\partial}{\partial x}[\gamma f]$ two ways

$$\frac{\partial}{\partial x}[\gamma f] = \gamma\frac{\partial f}{\partial x} + f\frac{\partial \gamma}{\partial f} = \gamma_{i,k}\,\frac{(f_{i+1,k} - f_{i-1,k})}{2\Delta} + f_{i,k}\,\frac{(\gamma_{i+1,k} - \gamma_{i-1,k})}{2\Delta}$$

$$\frac{\partial}{\partial x}[\gamma f] = \frac{\gamma_{i+1,k}\, f_{i+1,k} - \gamma_{i-1,k}\, f_{i-1,k}}{2\Delta}$$

So we average the two

$$\frac{\partial}{\partial x}[\gamma f] = \frac{1}{2}\left[\frac{\gamma_{i+1,k}\, f_{i+1,k} - \gamma_{i-1,k}\, f_{i-1,k}}{2\Delta} + \gamma_{i,k}\,\frac{(f_{i+1,k} - f_{i-1,k})}{2\Delta} + f_{i,k}\frac{(\gamma_{i+1,k} - \gamma_{i-1,k})}{2\Delta}\right]$$

$$= \frac{1}{2}\left[\frac{(\gamma_{i+1,k} + \gamma_{i,k})f_{i+1,k} - (\gamma_{i,k} + \gamma_{i-1,k})f_{i-1,k}}{2\Delta} + f_{i,k}\,\frac{(\gamma_{i+1,k} - \gamma_{i-1,k})}{2\Delta}\right]$$

$$= \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\, f_{i+1,k} - \gamma_{i-1/2\,,k}\, f_{i-1,k} + f_{i,k}\frac{(\gamma_{i+1,k} - \gamma_{i-1,k} + (\gamma_{i,k} - \gamma_{i,k}))}{2}\right]$$

$$= \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\, f_{i+1,k} - \gamma_{i-1/2\,,k}\, f_{i-1,k} + f_{i,k}(\gamma_{i+1/2\,,k} - \gamma_{i-1/2\,,k})\right]$$

$$= \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\left(f_{i+1,k} + f_{i,k}\right) - \gamma_{i-1/2\,,k}\left(f_{i-1,k} + f_{i,k}\right)\right]$$

Finally we arrive at

$$\frac{\partial}{\partial x}[\gamma f] = \frac{1}{2\Delta}\left[\gamma_{i+1/2\,,k}\left(u_{i+1,k} + u_{i,k}\right) - \gamma_{i-1/2\,,k}\left(u_{i,k} + u_{i-1,k}\right)\right]$$

1.2.6 Forms for the Finite Differences in u

For our finite differences we will use a mass lumping technique on the intertial term (Marfurt, 1984). Recalling our non rotated equation of motion in u given by (1.13a)

$$(\mu p_y^2-\rho)\omega^2 u = R_{2x}(\lambda+2\mu)\frac{\partial u}{\partial x}+R_{1x}^2\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x}\right]+R_{1x}R_{1z}\left(\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right]+\frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]\right) + R_{2z}\mu\frac{\partial u}{\partial z}+R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial u}{\partial z}\right]-R_{1x}i\omega p_y\left(\frac{\partial}{\partial x}\left[\lambda v\right]+\mu\frac{\partial v}{\partial x}\right)$$

If we gather the forms from Appendix A.1 we obtain

$$\frac{A5UU}{u_{i,k}} \Rightarrow b\omega^2\rho' - \frac{R_{1x}^2}{\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k}+(\lambda+2\mu)_{i-1/2\,,k}\right]-\frac{R_{1z}^2}{\Delta^2}\left[\mu_{i,k+1/2}+\mu_{i,k-1/2}\right] + \frac{R_{2z}}{2\Delta}\left[\mu_{i,k-1/2}-\mu_{i,k+1/2}\right]+\frac{R_{2x}}{2\Delta}\left[(\lambda+2\mu)_{i-1/2\,,k}-(\lambda+2\mu)_{i+1/2\,,k}\right] \quad (1.20a)$$

$$\frac{A5UV}{v_{i,k}} \Rightarrow R_{1x}\frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i-1/2\,,k}-(\lambda-\mu)_{i+1/2\,,k}\right] \quad (1.20b)$$

$$\frac{A8UU}{u_{i+1,k}} \Rightarrow \frac{(1-b)}{4}\omega^2\rho' + \left(\frac{R_{1x}^2}{\Delta^2}+\frac{R_{2x}}{2\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k} \quad (1.20c)$$

$$\frac{A8UV}{v_{i+1,k}} \Rightarrow -R_{1x}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i+1/2\,,k} \quad (1.20d)$$

$$\frac{A2UU}{u_{i-1,k}} \Rightarrow \frac{(1-b)}{4}\omega^2\rho' + \left(\frac{R_{1x}^2}{\Delta^2}-\frac{R_{2x}}{2\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k} \quad (1.20e)$$

$$\frac{A2UV}{v_{i-1,k}} \Rightarrow R_{1x}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i-1/2\,,k} \quad (1.20f)$$

$$\frac{A6UU}{u_{i,k+1}} \Rightarrow \frac{(1-b)}{4}\omega^2\rho' + \left(\frac{R_{1z}^2}{\Delta^2}+\frac{R_{2z}}{2\Delta}\right)\mu_{i\,,k+1/2} \quad (1.20g)$$

$$\frac{A4UU}{u_{i,k-1}} \Rightarrow \frac{(1-b)}{4}\omega^2\rho' + \left(\frac{R_{1z}^2}{\Delta^2}-\frac{R_{2z}}{2\Delta}\right)\mu_{i\,,k-1/2} \quad (1.20h)$$

$$\frac{A9UW}{w_{i+1,k+1}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i+1,k}+\mu_{i,k+1}) \quad (1.20i)$$

$$\frac{A7UW}{w_{i+1,k-1}} \Rightarrow -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i+1,k}+\mu_{i,k-1}) \tag{1.20j}$$

$$\frac{A3UW}{w_{i-1,k+1}} \Rightarrow -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i-1,k}+\mu_{i,k+1}) \tag{1.20k}$$

$$\frac{A1UW}{w_{i-1,k-1}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i-1,k}+\mu_{i,k-1}) \tag{1.20l}$$

The rotated terms are from (1.15a)

$$\begin{aligned}(\mu p_y^2-\rho)\omega^2 u = {} & \frac{R_{2x}}{\sqrt{2}}(\lambda+2\mu)\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)+\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)(\lambda+2\mu)\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\\ & +\frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\lambda\left(\frac{\partial w}{\partial x'}+\frac{\partial w}{\partial z'}\right)\right]+\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right]\right\}\\ & +\frac{R_{2z}}{\sqrt{2}}\left\{\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right\}+\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\mu\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right]\\ & -R_{1x}i\frac{\omega p_y}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\lambda v]-\frac{\partial}{\partial z'}[\lambda v]+\mu\left(\frac{\partial v}{\partial x'}-\frac{\partial v}{\partial z'}\right)\right\}\end{aligned}$$

Collection of the rotated forms in Appendix A.1 yields. Before proceeding, recall that each term is pre-multiplied by the weighting $(1-a)$.

$$\begin{aligned}&\frac{A5UU}{u_{i,k}} \Rightarrow\\ &\frac{R_{2x}}{4\Delta}\left[(\lambda+2\mu)_{i-1/2\,,k-1/2}+(\lambda+2\mu)_{i-1/2\,,k+1/2}-(\lambda+2\mu)_{i+1/2\,,k+1/2}-(\lambda+2\mu)_{i+1/2\,,k-1/2}\right]\\ &-\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}+(\lambda+2\mu)_{i-1/2\,,k-1/2}+(\lambda+2\mu)_{i-1/2\,,k+1/2}+(\lambda+2\mu)_{i+1/2\,,k-1/2}\right]\\ &+\frac{R_{2z}}{4\Delta}\left[\mu_{i+1/2\,,k-1/2}+\mu_{i-1/2\,,k-1/2}-\mu_{i-1/2\,,k+1/2}-\mu_{i+1/2\,,k+1/2}\right]\\ &+\frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}+\mu_{i+1/2\,,k-1/2}+\mu_{i+1/2\,,k+1/2}+\mu_{i-1/2\,,k-1/2}\right]\end{aligned} \tag{1.21a}$$

$$\frac{A9UU}{u_{i+1,k+1}} \Rightarrow \left(\frac{R_{1x}^2}{4\Delta^2}+\frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k+1/2}+\left(\frac{R_{1z}^2}{4\Delta^2}+\frac{R_{2z}}{4\Delta}\right)\mu_{i+1/2\,,k+1/2} \tag{1.21b}$$

$$\frac{A1UU}{u_{i-1,k-1}} \Rightarrow \left(\frac{R_{1x}^2}{4\Delta^2}-\frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k-1/2}+\left(\frac{R_{1z}^2}{4\Delta^2}-\frac{R_{2z}}{4\Delta}\right)\mu_{i-1/2\,,k-1/2} \tag{1.21c}$$

$$\frac{A3UU}{u_{i-1,k+1}} \Rightarrow \left(\frac{R_{1x}^2}{4\Delta^2}-\frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k+1/2}+\left(\frac{R_{1z}^2}{4\Delta^2}+\frac{R_{2z}}{4\Delta}\right)\mu_{i-1/2\,,k+1/2} \tag{1.21d}$$

$$\frac{A7UU}{u_{i+1,k-1}} \Rightarrow \left(\frac{R_{1x}^2}{4\Delta^2} + \frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k-1/2} + \left(\frac{R_{1z}^2}{4\Delta^2} - \frac{R_{2z}}{4\Delta}\right)\mu_{i+1/2\,,k-1/2} \quad (1.21e)$$

$$\frac{A6UU}{u_{i,k+1}} \Rightarrow -\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2} + (\lambda+2\mu)_{i-1/2\,,k+1/2}\right] + \frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2} + \mu_{i-1/2\,,k+1/2}\right] \quad (1.21f)$$

$$\frac{A8UU}{u_{i+1,k}} \Rightarrow \frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2} + (\lambda+2\mu)_{i+1/2\,,k-1/2}\right] - \frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2} + \mu_{i+1/2\,,k-1/2}\right] \quad (1.21g)$$

$$\frac{A2UU}{u_{i-1,k}} \Rightarrow -\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2} + (\lambda+2\mu)_{i-1/2\,,k-1/2}\right] - \frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k-1/2} + \mu_{i-1/2\,,k+1/2}\right] \quad (1.21h)$$

$$\frac{A4UU}{u_{i,k-1}} \Rightarrow -\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k-1/2} + (\lambda+2\mu)_{i+1/2\,,k-1/2}\right] + \frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k-1/2} + \mu_{i+1/2\,,k-1/2}\right] \quad (1.21i)$$

$$\frac{A5UW}{w_{i,k}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}$$
$$\left[(\lambda+\mu)_{i+1/2\,,k-1/2} + (\lambda+\mu)_{i-1/2\,,k+1/2} - (\lambda+\mu)_{i+1/2\,,k+1/2} - (\lambda+\mu)_{i-1/2\,,k-1/2}\right] \quad (1.21j)$$

$$\frac{A6UW}{w_{i,k+1}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k+1/2}\right] \quad (1.21k)$$

$$\frac{A8UW}{w_{i+1,k}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2}\right] \quad (1.21l)$$

$$\frac{A2UW}{w_{i-1,k}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k-1/2}\right] \quad (1.21m)$$

$$\frac{A4UW}{w_{i,k-1}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{1.21n}$$

$$\frac{A9UW}{w_{i+1,k+1}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{1.21o}$$

$$\frac{A1UW}{w_{i-1,k-1}} \Rightarrow \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{1.21p}$$

$$\frac{A3UW}{w_{i-1,k+1}} \Rightarrow -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{1.21q}$$

$$\frac{A7UW}{w_{i+1,k-1}} \Rightarrow -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{1.21r}$$

$$\frac{A5UV}{v_{i,k}} \Rightarrow \frac{R_{1x}i\omega p_y}{4\Delta}$$
$$\left[(\lambda-\mu)_{i-1/2\,,k-1/2} + (\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{1.21s}$$

$$\frac{A9UV}{v_{i+1,k+1}} \Rightarrow -\frac{R_{1x}i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{1.21t}$$

$$\frac{A1UV}{v_{i-1,k-1}} \Rightarrow \frac{R_{1x}i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{1.21u}$$

$$\frac{A3UV}{v_{i-1,k+1}} \Rightarrow \frac{R_{1x}i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{1.21v}$$

$$\frac{A7UV}{v_{i+1,k-1}} \Rightarrow -\frac{R_{1x}i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{1.21w}$$

1.2.7 Forms for the Finite Differences in v

Recalling our general equation of motion in the non-rotated from (1.13b)

$$-(p_y^2(\lambda+2\mu)-\rho)\omega^2 v + R_{2x}\left(\mu\frac{\partial v}{\partial x}\right) + R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right] + R_{2z}\mu\frac{\partial v}{\partial z} + R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial v}{\partial z}\right] - i\omega p_y\left(R_{1x}\left\{\frac{\partial}{\partial x}\left[\mu u\right] + \lambda\frac{\partial u}{\partial x}\right\} + R_{1z}\left\{\frac{\partial}{\partial z}\left[\mu w\right] + \lambda\frac{\partial w}{\partial z}\right\}\right) = f$$

we discretize by collecting the non rotated finite difference forms in Appendix A.2

$$\frac{A5VV}{v_{i,k}} = b\omega^2\rho'' - \frac{R_{1z}^2}{\Delta^2}[\mu_{i\,,k+1/2} + \mu_{i\,,k-1/2}] - \frac{R_{1x}^2}{\Delta^2}[\mu_{i+1/2\,,k} + \mu_{i-1/2\,,k}] + \frac{R_{2x}}{2\Delta}[\mu_{i-1/2\,,k} - \mu_{i+1/2\,,k}] + \frac{R_{2z}}{2\Delta}[\mu_{i\,,k-1/2} - \mu_{i\,,k+1/2}] \quad (1.22a)$$

$$\frac{A6VV}{v_{i,k+1}} = \frac{(1-b)}{4}\omega^2\rho'' + \left(\frac{R_{1z}^2}{\Delta^2} + \frac{R_{2z}}{2\Delta}\right)\mu_{i\,,k+1/2} \quad (1.22b)$$

$$\frac{A4VV}{v_{i,k-1}} = \frac{(1-b)}{4}\omega^2\rho'' + \left(\frac{R_{1z}^2}{\Delta^2} - \frac{R_{2z}}{2\Delta}\right)\mu_{i\,,k-1/2} \quad (1.22c)$$

$$\frac{A8VV}{v_{i+1,k}} = \frac{(1-b)}{4}\omega^2\rho'' + \left(\frac{R_{1x}^2}{\Delta^2} + \frac{R_{2x}}{2\Delta}\right)\mu_{i+1/2\,,k} \quad (1.22d)$$

$$\frac{A2VV}{v_{i-1,k}} = \frac{(1-b)}{4}\omega^2\rho'' + \left(\frac{R_{1x}^2}{\Delta^2} - \frac{R_{2x}}{2\Delta}\right)\mu_{i-1/2\,,k} \quad (1.22e)$$

$$\frac{A5VW}{w_{i,k}} = R_{1z}\frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i\,,k+1/2} - (\lambda-\mu)_{i\,,k-1/2}\right] \quad (1.22f)$$

$$\frac{A6VW}{w_{i,k+1}} = -R_{1z}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k+1/2} \quad (1.22g)$$

$$\frac{A4VW}{w_{i,k-1}} = R_{1z}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k-1/2} \quad (1.22h)$$

$$\frac{A5VU}{u_{i,k}} = R_{1x}\frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i+1/2\,,k} - (\lambda-\mu)_{i-1/2\,,k}\right] \quad (1.22i)$$

$$\frac{A8VU}{u_{i+1,k}} = -R_{1x}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i+1/2\,,k} \quad (1.22j)$$

$$\frac{A2VU}{u_{i-1,k}} = R_{1x}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i-1/2\,,k} \quad (1.22k)$$

For our rotated equation of motion for v given by (1.15b)

$$
\begin{aligned}
(p_y^2(\lambda+2\mu)-\rho)\omega^2 v &= \frac{\mu}{\sqrt{2}}\left(\frac{\partial v}{\partial x'}(R_{2x}+R_{2z})+\frac{\partial v}{\partial z'}(R_{2z}-R_{2x})\right)\\
\frac{1}{2}\left[(R_{1x}^2+R_{1z}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial x'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial z'}\right]\right)+(R_{1z}^2-R_{1x}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial x'}\right]\right)\right]&\\
-i\omega p_y\frac{R_{1x}}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\mu u]-\frac{\partial}{\partial z'}[\mu u]+\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right\}&\\
-i\omega p_y\frac{R_{1z}}{\sqrt{2}}\left\{\frac{\partial}{\partial z'}[\mu w]+\frac{\partial}{\partial x'}[\mu w]+\lambda\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right\}&
\end{aligned}
$$

we collect the finite difference forms from Appendix A.2

$$
\begin{aligned}
&\frac{A5VV}{v_{i,k}}\Rightarrow\\
&\frac{(R_{2x}+R_{2z})}{4\Delta}[\mu_{i-1/2\,,k-1/2}\,-\mu_{i+1/2\,,k+1/2}]+\frac{(R_{2z}-R_{2x})}{4\Delta}[\mu_{i+1/2\,,k-1/2}\,-\mu_{i-1/2\,,k+1/2}]\\
&-\frac{(R_{1x}^2+R_{1z}^2)}{4\Delta}[\mu_{i+1/2\,,k+1/2}\,+\mu_{i-1/2\,,k-1/2}]+\frac{(R_{1z}^2+R_{1x}^2)}{4\Delta}[\mu_{i-1/2\,,k+1/2}\,+\mu_{i+1/2\,,k-1/2}]
\end{aligned}
\tag{1.23a}
$$

$$
\frac{A9VV}{v_{i+1,k+1}}\Rightarrow\left(\frac{R_{1x}^2+R_{1z}^2}{4\Delta^2}+\frac{R_{2x}+R_{2z}}{4\Delta}\right)\mu_{i+1/2\,,k+1/2} \tag{1.23b}
$$

$$
\frac{A1VV}{v_{i-1,k-1}}\Rightarrow\left(\frac{R_{1x}^2+R_{1z}^2}{4\Delta^2}-\frac{R_{2x}+R_{2z}}{4\Delta}\right)\mu_{i-1/2\,,k-1/2} \tag{1.23c}
$$

$$
\frac{A3VV}{v_{i-1,k+1}}\Rightarrow\left(\frac{R_{1z}^2+R_{1x}^2}{4\Delta^2}+\frac{R_{2x}-R_{2z}}{4\Delta}\right)\mu_{i-1/2\,,k+1/2} \tag{1.23d}
$$

$$
\frac{A7VV}{v_{i+1,k-1}}\Rightarrow\left(\frac{R_{1z}^2+R_{1x}^2}{4\Delta^2}-\frac{R_{2x}-R_{2z}}{4\Delta}\right)\mu_{i+1/2\,,k-1/2} \tag{1.23e}
$$

$$
\frac{A6VV}{v_{i,k+1}}\Rightarrow\frac{(R_{1z}^2-R_{1x}^2)}{4\Delta^2}\left(\mu_{i+1/2\,,k+1/2}\,+\mu_{i-1/2\,,k+1/2}\right) \tag{1.23f}
$$

$$
\frac{A8VV}{v_{i+1,k}}\Rightarrow-\frac{(R_{1z}^2-R_{1x}^2)}{4\Delta^2}\left(\mu_{i+1/2\,,k+1/2}\,+\mu_{i+1/2\,,k-1/2}\right) \tag{1.23g}
$$

$$
\frac{A2VV}{v_{i-1,k}}\Rightarrow-\frac{(R_{1z}^2-R_{1x}^2)}{4\Delta^2}\left(\mu_{i-1/2\,,k-1/2}\,+\mu_{i-1/2\,,k+1/2}\right) \tag{1.23h}
$$

$$\frac{A4VV}{v_{i,k-1}} \Rightarrow \frac{(R_{1z}^2 - R_{1x}^2)}{4\Delta^2}\left(\mu_{i-1/2\,,k-1/2} + \mu_{i+1/2\,,k-1/2}\right) \tag{1.23i}$$

$$\frac{A5VU}{u_{i,k}} \Rightarrow -\frac{R_{1x} i\omega p_y}{4\Delta}$$
$$\left[(\lambda-\mu)_{i-1/2\,,k-1/2} + (\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{1.23j}$$

$$\frac{A9VU}{u_{i+1,k+1}} \Rightarrow -\frac{R_{1x} i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{1.23k}$$

$$\frac{A1VU}{u_{i-1,k-1}} \Rightarrow \frac{R_{1x} i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{1.23l}$$

$$\frac{A3VU}{u_{i-1,k+1}} \Rightarrow \frac{R_{1x} i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{1.23m}$$

$$\frac{A7VU}{u_{i+1,k-1}} \Rightarrow -\frac{R_{1x} i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{1.23n}$$

$$\frac{A5VW}{w_{i,k}} \Rightarrow$$
$$\frac{R_{1z} i\omega p_y}{4\Delta}\left[(\lambda-\mu)_{i+1/2\,,k+1/2} + (\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{1.23o}$$

$$\frac{A9VW}{w_{i+1,k+1}} \Rightarrow -\frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{1.23p}$$

$$\frac{A1VW}{w_{i-1,k-1}} \Rightarrow \frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{1.23q}$$

$$\frac{A3VW}{w_{i-1,k+1}} \Rightarrow \frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{1.23r}$$

$$\frac{A7VW}{w_{i+1,k-1}} \Rightarrow -\frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{1.23s}$$

1.2.8 Forms for the Finite Differences in w

We recall our continuous non rotated equation of motion in w given by (1.13c)

$$-(\mu p_y^2-\rho)\omega^2 w+R_{2z}(\lambda+2\mu)\frac{\partial w}{\partial z}+R_{1z}^2\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z}\right]+R_{1x}R_{1z}\left(\frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right]\right)$$
$$+R_{2x}\left(\mu\frac{\partial w}{\partial x}\right)+R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial w}{\partial x}\right]-R_{1z}i\omega p_y\left(\frac{\partial}{\partial z}\left[\lambda v\right]+\mu\frac{\partial v}{\partial z}\right)$$

We collect terms from Appendix A.3 and find

$$\frac{A5WW}{w_{i,k}}=b\omega^2\rho'$$
$$-\frac{R_{1z}^2}{\Delta^2}\left[(\lambda+2\mu)_{i\,,k+1/2}\,+(\lambda+2\mu)_{i,\,,k-1/2}\right]-\frac{R_{1x}^2}{\Delta^2}\left[\mu_{i+1/2\,,k}\,+\mu_{i-1/2\,,k}\right]$$
$$+\frac{R_{2x}}{2\Delta}\left[\mu_{i-1/2\,,k}\,-\mu_{i+1/2\,,k}\right]+\frac{R_{2z}}{2\Delta}\left[(\lambda+2\mu)_{i\,,k-1/2}\,-(\lambda+2\mu)_{i\,,k+1/2}\right] \quad (1.24a)$$

$$\frac{A5WV}{v_{i,k}}=R_{1z}\frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i\,,k-1/2}\,-(\lambda-\mu)_{i\,,k+1/2}\right] \quad (1.24b)$$

$$\frac{A6WW}{w_{i,k+1}}=\frac{(1-b)}{4}\omega^2\rho'+\left(\frac{R_{1z}^2}{\Delta^2}+\frac{R_{2z}}{2\Delta}\right)(\lambda+2\mu)_{i\,,k+1/2} \quad (1.24c)$$

$$\frac{A6WV}{v_{i,k+1}}=-R_{1z}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k+1/2} \quad (1.24d)$$

$$\frac{A4WW}{w_{i,k-1}}=\frac{(1-b)}{4}\omega^2\rho'+\left(\frac{R_{1z}^2}{\Delta^2}-\frac{R_{2z}}{2\Delta}\right)(\lambda+2\mu)_{i\,,k-1/2} \quad (1.24e)$$

$$\frac{A4WV}{v_{i,k-1}}=R_{1z}\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k-1/2} \quad (1.24f)$$

$$\frac{A8WW}{w_{i+1,k}}=\frac{(1-b)}{4}\omega^2\rho'+\left(\frac{R_{1x}^2}{\Delta^2}+\frac{R_{2x}}{2\Delta}\right)\mu_{i+1/2\,,k} \quad (1.24g)$$

$$\frac{A2WW}{w_{i-1,k}}=\frac{(1-b)}{4}\omega^2\rho'+\left(\frac{R_{1x}^2}{\Delta^2}-\frac{R_{2x}}{2\Delta}\right)\mu_{i-1/2\,,k} \quad (1.24h)$$

$$\frac{A9WU}{u_{i+1\,,k+1}}=\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i\,,k+1}\,+\mu_{i+1\,,k}) \quad (1.24i)$$

$$\frac{A7WU}{u_{i+1\,,k-1}}=-\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i\,,k-1}\,+\mu_{i+1\,,k}) \quad (1.24j)$$

$$\frac{A3WU}{u_{i-1\,,k+1}} = -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i\,,k+1} + \mu_{i-1\,,k}) \tag{1.24k}$$

$$\frac{A1WU}{u_{i-1\,,k-1}} = \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda_{i\,,k-1} + \mu_{i-1\,,k}) \tag{1.24l}$$

Then recalling our rotated continuous equation of motion in w given by (1.15c)

$$\begin{aligned}(\mu p_y^2 - \rho)\omega^2 w = {} & \frac{R_{2z}}{\sqrt{2}}(\lambda+2\mu)\left(\frac{\partial w}{\partial z'} + \frac{\partial w}{\partial x'}\right) + \frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'} + \frac{\partial}{\partial x'}\right)\left[(\lambda+2\mu)\left(\frac{\partial w}{\partial z'} + \frac{\partial w}{\partial x'}\right)\right] \\ & + \frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial z'} + \frac{\partial}{\partial x'}\right)\left[\lambda\left(\frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'}\right)\right] + \left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial u}{\partial z'} + \frac{\partial u}{\partial x'}\right)\right]\right\} \\ & + \frac{R_{2x}}{\sqrt{2}}\left\{\mu\left(\frac{\partial w}{\partial x'} - \frac{\partial w}{\partial z'}\right)\right\} + \frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial z'}\right)\left[\mu\frac{\partial w}{\partial x'} - \frac{\partial w}{\partial z'}\right] \\ & - R_{1z}i\frac{\omega p_y}{\sqrt{2}}\left\{\mu\left(\frac{\partial v}{\partial z'} + \frac{\partial v}{\partial x'}\right) + \frac{\partial}{\partial z'}[\lambda v] + \frac{\partial}{\partial x'}[\lambda v]\right\}\end{aligned}$$

we can collect our forms from Appendix A.3 and find

$$\begin{aligned}&\frac{A5WW}{w_{i,k}} \\ &= \frac{R_{2z}}{4\Delta}\left[(\lambda+2\mu)_{i-1/2\,,k-1/2} + (\lambda+2\mu)_{i+1/2\,,k-1/2} - (\lambda+2\mu)_{i+1/2\,,k+1/2} - (\lambda+2\mu)_{i-1/2\,,k+1/2}\right] \\ &\qquad + \frac{R_{2x}}{4\Delta}\left[\mu_{i-1/2\,,k+1/2} + \mu_{i-1/2\,,k-1/2} - \mu_{i+1/2\,,k-1/2} - \mu_{i+1/2\,,k+1/2}\right] \\ &- \frac{R_{1z}^2}{4\Delta}\left[(\lambda+2\mu)_{i-1/2\,,k-1/2} + (\lambda+2\mu)_{i+1/2\,,k+1/2} + (\lambda+2\mu)_{i-1/2\,,k+1/2} + (\lambda+2\mu)_{i+1/2\,,k-1/2}\right] \\ &\qquad - \frac{R_{1x}^2}{4\Delta}\left[\mu_{i+1/2\,,k-1/2} + \mu_{i-1/2\,,k-1/2} + \mu_{i+1/2\,,k+1/2} + \mu_{i-1/2\,,k+1/2}\right]\end{aligned} \tag{1.25a}$$

$$\frac{A9WW}{w_{i+1,k+1}} = \left(\frac{R_{1z}^2}{4\Delta^2} + \frac{R_{2z}}{4\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k+1/2} + \left(\frac{R_{1x}^2}{4\Delta^2} + \frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k+1/2} \tag{1.25b}$$

$$\frac{A1WW}{w_{i-1,k-1}} = \left(\frac{R_{1z}^2}{4\Delta^2} - \frac{R_{2z}}{4\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k-1/2} + \left(\frac{R_{1x}^2}{4\Delta^2} - \frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k-1/2} \tag{1.25c}$$

$$\frac{A3WW}{w_{i-1,k+1}} = \left(\frac{R_{1z}^2}{4\Delta^2} + \frac{R_{2z}}{4\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k+1/2} + \left(\frac{R_{1x}^2}{4\Delta^2} - \frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i-1/2\,,k+1/2} \tag{1.25d}$$

$$\frac{A7WW}{w_{i+1,k-1}} = \left(\frac{R_{1z}^2}{4\Delta^2} - \frac{R_{2z}}{4\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k-1/2} + \left(\frac{R_{1x}^2}{4\Delta^2} + \frac{R_{2x}}{4\Delta}\right)(\lambda+2\mu)_{i+1/2\,,k-1/2} \tag{1.25e}$$

$$\frac{A6WW}{w_{i,k+1}} = \frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\ +(\lambda+2\mu)_{i-1/2\,,k+1/2}\right] - \frac{R_{1x}^2}{4\Delta^2}(\mu_{i-1/2\,,k+1/2}+\mu_{i+1/2\,,k+1/2}) \tag{1.25f}$$

$$\frac{A8WW}{w_{i+1,k}} = -\frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\ +(\lambda+2\mu)_{i+1/2\,,k-1/2}\right] + \frac{R_{1x}^2}{4\Delta^2}(\mu_{i+1/2\,,k-1/2}+\mu_{i+1/2\,,k+1/2}) \tag{1.25g}$$

$$\frac{A2WW}{w_{i-1,k}} = -\frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k-1/2}\ +(\lambda+2\mu)_{i-1/2\,,k+1/2}\right] + \frac{R_{1x}^2}{4\Delta^2}(\mu_{i-1/2\,,k+1/2}+\mu_{i-1/2\,,k-1/2}) \tag{1.25h}$$

$$\frac{A4WW}{w_{i,k-1}} = \frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k-1/2}\ +(\lambda+2\mu)_{i-1/2\,,k-1/2}\right] - \frac{R_{1x}^2}{4\Delta^2}(\mu_{i+1/2\,,k-1/2}+\mu_{i-1/2\,,k-1/2}) \tag{1.25i}$$

$$\frac{A5WU}{u_{i,k}} = \frac{R_{1x}R_{1z}}{4\Delta^2}$$
$$\left[(\lambda+\mu)_{i-1/2\,,k+1/2}\ +(\lambda+\mu)_{i+1/2\,,k-1/2}\ -(\lambda+\mu)_{i+1/2\,,k+1/2}\ -(\lambda+\mu)_{i-1/2\,,k-1/2}\right] \tag{1.25j}$$

$$\frac{A8WU}{u_{i+1,k}} = \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k+1/2}\ -(\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{1.25k}$$

$$\frac{A6WU}{u_{i,k+1}} = \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k+1/2}\ -(\lambda-\mu)_{i+1/2\,,k+1/2}\right] \tag{1.25l}$$

$$\frac{A2WU}{u_{i-1,k}} = \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k-1/2}\ -(\lambda-\mu)_{i-1/2\,,k+1/2}\right] \tag{1.25m}$$

$$\frac{A4WU}{u_{i,k-1}} = \frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k-1/2}\ -(\lambda-\mu)_{i-1/2\,,k-1/2}\right] \tag{1.25n}$$

$$\frac{A9WU}{u_{i+1,k+1}} = \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{1.25o}$$

$$\frac{A1WU}{u_{i-1,k-1}} = \frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{1.25p}$$

$$\frac{A3WU}{u_{i-1,k+1}} = -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{1.25q}$$

$$\frac{A7WU}{u_{i+1,k-1}} = -\frac{R_{1x}R_{1z}}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{1.25r}$$

$$\frac{A5WV}{v_{i,k}} = \frac{R_{1z} i\omega p_y}{4\Delta}$$

$$\left[(\lambda-\mu)_{i+1/2\,,k-1/2} + (\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k+1/2}\right] \tag{1.25s}$$

$$\frac{A9WV}{v_{i+1,k+1}} = -\frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{1.25t}$$

$$\frac{A1WV}{v_{i-1,k-1}} = \frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{1.25u}$$

$$\frac{A3WV}{v_{i-1,k+1}} = -\frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{1.25v}$$

$$\frac{A7WV}{v_{i+1,k-1}} = \frac{R_{1z} i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{1.25w}$$

CHAPTER 2
Total Field Scattered Field Approach for Plane Waves

Following from Roecker, Baker, and McLaughlin manuscript in prepartion (2009), we develop two methods for specifying an analytic solution. Our first approach, which we will not ultimately use, comes from considering the sourceless problem

$$S\mathbf{u} = 0 \tag{2.1}$$

which is sensible since our teleseismic sources will be out of the computational domain. Though 2.1 is ill-posed, we can still look for a distribution of sources $\mathbf{f}_0$, that will generate a desired $\mathbf{u} = \mathbf{u}_0$ in a simple medium discretized by S_0, i.e.

$$S_0\mathbf{u}_0 = \mathbf{f}_0 \tag{2.2}$$

As consequence of the work done in developing an analytic solution we have a means of specifying a $\mathbf{u}_0$ for a simple model and can certainly generate S_0 via finite differences. Direct multiplication in 2.2 will gives a force distribution $\mathbf{f}_0$, such that we can solve the heterogeneous problem $S\mathbf{u} = \mathbf{f}_0$ for $\mathbf{u}$. However, the differencing nature of S_0 means that most of the energy will be located in the location of steep velocity contrast, which tend to be near the free surface. Consequently, our free surface sources can be several orders of magnitude greater than that of sources within the medium. Upon implementation we are now heavily reliant upon our implementation of free surface boundary conditions, which leads to numerical instability.

An alternative approach would be to decompose the total wavefield, $\mathbf{u}_t$, into its background wavefield, $\mathbf{u}_0$, and it's perturbed wavefield, $\mathbf{u}_h$,

$$\mathbf{u}_t = \mathbf{u}_0 + \mathbf{u}_h \tag{2.3}$$

In a similar fashion we can also decompose our differencing matrix S

$$S = S_0 + S_h \tag{2.4}$$

Then we write our sourceless problem 2.1

$$S\mathbf{u}_t = (S_0 + S_h)(\mathbf{u}_0 + \mathbf{u}_h) = S_0\mathbf{u}_0 + S_h\mathbf{u}_0 + S\mathbf{u}_h = 0 \tag{2.5}$$

Recalling the background plane wave problem has no internal forces, $S_0\mathbf{u}_0 = 0$ we can rearrange the last equality of 2.5 into

$$S\mathbf{u}_h = -S_h\mathbf{u}_0 = -(S - S_0)\mathbf{u}_0 \tag{2.6}$$

Thus

$$\mathbf{u_h} = -S^{-1}(S - S_0)\mathbf{u}_0$$

and expanding on $\mathbf{u_h}$ gives us a desired expression for our scattered field,

$$\mathbf{u_t} = \mathbf{u}_0 - S^{-1}(S - S_0)\mathbf{u}_0 \tag{2.7}$$

Now we identity the source distribution as $\mathbf{f} = -(S - S_0)\mathbf{u}_0$, indicating the forces now only appear at perturbations in the background model, which generally will not include the boundaries, and we circumvent the numerical limitation of the previous approach.

To demonstrate the approaches are mathematically equivalent we recall that in the total field approach

$$S\mathbf{u}_t = S_0\mathbf{u}_0 = \mathbf{f}_0$$

and from 2.5

$$S\mathbf{u}_t = (S_0 + S_h)(\mathbf{u}_0 + \mathbf{u}_h) = \mathbf{f}_0 + S_h\mathbf{u}_0 + S_h\mathbf{u}_0 + S\mathbf{u}_h = \mathbf{f}_0$$

and finally

$$S\mathbf{u_h} = -S_h\mathbf{u}_0 \tag{2.8}$$

which is indeed equation 2.6. Hence we can either obtain a total wavefield by considering the distribution of sources that create a wavefield in the background medium, or obtain a the scattered wavefield generated from sources located at medium perturbations. As previously noted for numerical reasons we prefer 2.6. When implementing we find our layer cake model by assuming an infinite extension of the heterogenous velocity model that corresponds to the angle of incidence. For example, if the wave comes from the 'left', our $1D$ model will be an extension of the values in the first column of our velocity matrix. For example see figure 2.1. Also, we specify our p_y velocity by choosing a velocity at the bottom of the model. This should represent our mantle velocity.

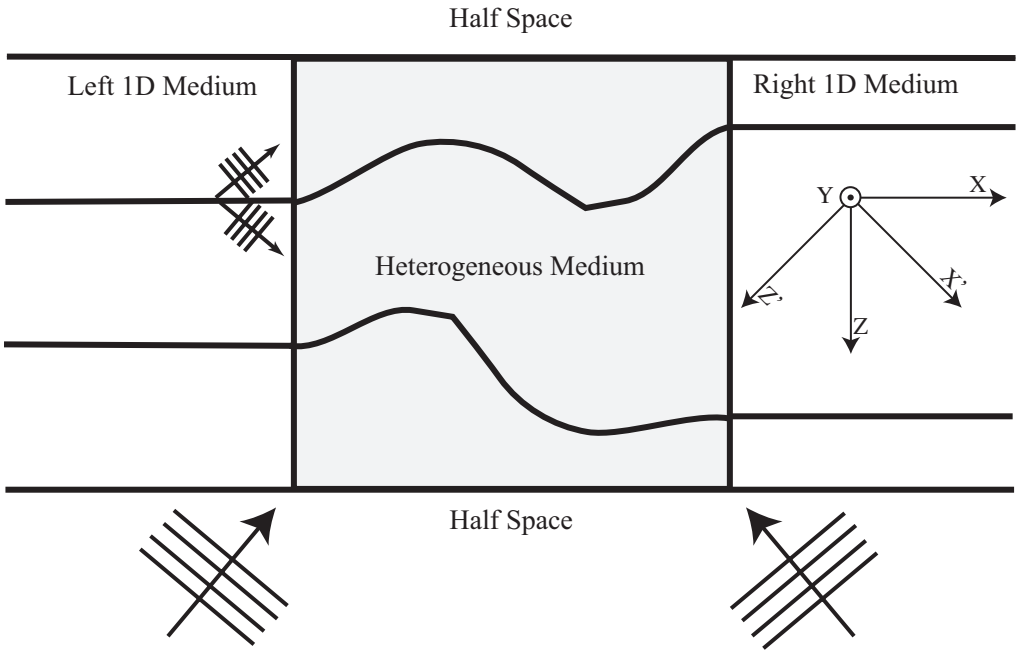

Figure 2.1: Schematic of the model. We consider a heterogenous medium (grey rectangle) bounded by half spaces on the top and bottom and vertically stratified media on either side. Plane waves enter the medium from below but converted waves may also enter the heterogeneous region from the side. The $1D$ media on the left and right are used for waves entering from the left and right, respectively, and are not necessarily the same. To prevent diffractions, the wave speeds in the heterogeneous medium match those in the $1D$ media at the boundaries. Orientations of the "non-rotated" (x, y, z) and "rotated" (x', y', z') axes are indicated on the right

CHAPTER 3
Analytic Solution to the 1D Problem

3.1 Solution to the Layer Problem

In developing the total field scattered field approach we require an analytic solution to the layer cake problem. A suitable approach is the Haskell propagator matrix technique. We follow the method prescribed for calculating the free surface response for incident P and SV waves Haskell (1953), Haskell 1962, as well as incident SH waves Haskell (1960), now referred to as H53, H62, and H60 respectively. In addition to what is provided directly from Haskell's work for the free surface, additionaly we require the displacement at every point in the medium, which can be found by 'backing out' the solution from the free surface.

3.1.1 Definitions

Our reference frame will be x positive right and z positive down as shown in 3.1. Following H53's notation (0) will relate to the free surface, 1, will represent the first layer, and (1) will denote the interface between the first and second layer etc. until the n'th layer.

The notation for this section is as follows

$$p = \text{Angular frequency}$$

$$\rho = \text{Density}$$

$$d = \text{Thickness}$$

$$k_x = \frac{p}{c} = \frac{2\pi}{\text{horizontal wavelength}} = \text{Angular wave number}$$

$$c = \text{Apparent wave speed}$$

$$u, w = \text{Displacement components in } x \text{ and } z \text{ directions}$$

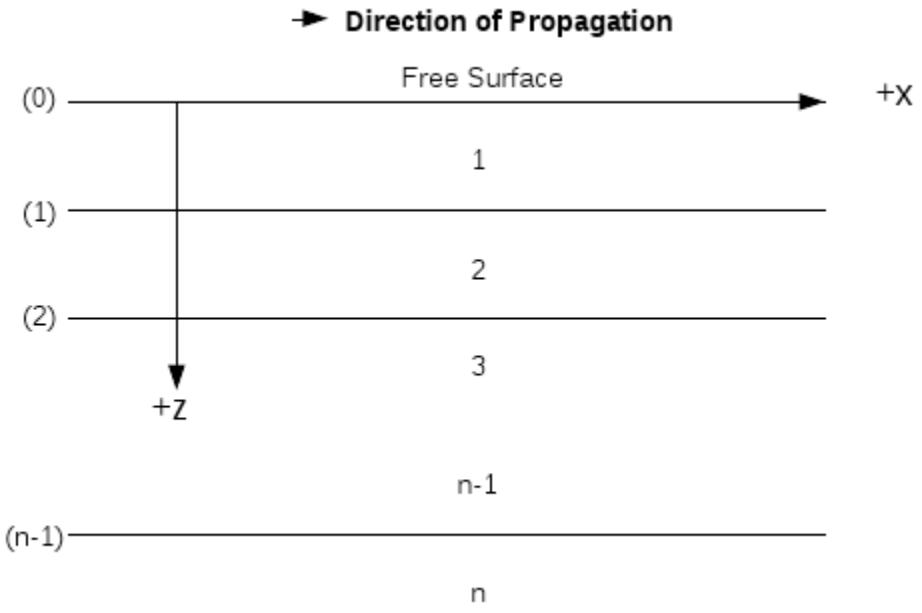

Figure 3.1: Direction of axes and number of layers and interfaces. Figure adapted from Haskell (1953)

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \text{Dilitational wave speed}$$

$$\beta = \sqrt{\frac{\mu}{\rho}} = \text{Rotational wave speed}$$

$$v = \alpha,\ \beta = \text{Velocity for } P \text{ or } S \text{ wave speed}$$

$$i = \text{Angle of incidence}$$

$$k_x = k \sin i = \text{Wave number in } x$$

$$k_z = k\cos i = \text{Wave number in } z$$

where

$$\sin i = \frac{v}{c}$$

We can now use

$$\cos^2 i + \sin^2 i = 1$$

or

$$\cos i = \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

Then

$$k_z = k_x \frac{\cos i}{\sin i} = k_x \frac{\sqrt{1 - \left(\frac{v}{c}\right)^2}}{\frac{v}{c}} = k_x \sqrt{\left(\frac{c}{v}\right)^2 - 1} = k_x r_v$$

To account for the exponential decay of potentials away from boundaries we have

$$r_v = \sqrt{\frac{c^2}{v^2} - 1} \qquad c > v$$

and

$$r_v = -i\sqrt{1 - \frac{c^2}{v^2}} \qquad c < v$$

$$\gamma = \frac{2\beta^2}{c^2}$$

$$\sigma = \text{Normal stress}$$

$$\tau = \text{Tangential stress}$$

We write our plane wave solution as

$$e^{i(pt - \mathbf{k}\cdot\mathbf{x})} = e^{ipt} e^{-i(k_x x + k_z z)} = e^{ipt} e^{-ik_x x} e^{-ik_x r_v z}$$

where in this context we mean $i = \sqrt{-1}$.

3.1.2 The Displacement Terms

We seek a solution to the homogenous wave equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla\times\nabla\times\mathbf{u} \tag{3.1}$$

The displacement in terms of potentials is given by

$$\mathbf{u} = \nabla\phi + \nabla\times\mathbf{\Psi}$$

or

$$u_x = u = \frac{\partial\Phi}{\partial x} - \frac{\partial\Psi}{\partial z} \tag{3.2a}$$

$$u_z = w = \frac{\partial\Phi}{\partial z} + \frac{\partial\Psi}{\partial x} \tag{3.2b}$$

Taking the divergence and curl of (3.1) gives

$$\frac{\partial^2\Phi}{\partial t^2} = \frac{(\lambda+2\mu)}{\rho}\nabla^2\Phi \tag{3.3a}$$

$$\frac{\partial^2\mathbf{\Psi}}{\partial t^2} = \frac{\mu}{\rho}\nabla^2\mathbf{\Psi} \tag{3.3b}$$

Now we define up our up going and down going trial solutions

$$\Phi_{\text{down}} = A_1 e^{i(pt-k_x x-k_z z)} = A_1 e^{i(pt-k_x x-k_x r_\alpha z)} \tag{3.4a}$$

$$\Phi_{\text{up}} = A_2 e^{i(pt-k_x x+k_z z)} = A_2 e^{i(pt-k_x x+k_x r_\alpha z)} \tag{3.4b}$$

$$\Psi_{\text{down}} = B_1 e^{i(pt-k_x x-k_z z)} = B_1 e^{i(pt-k_x x-k_x r_\beta z)} \tag{3.4c}$$

$$\Psi_{\text{up}} = B_2 e^{i(pt-k_x x+k_z z)} = B_2 e^{i(pt-k_x x+k_x r_\beta z)} \tag{3.4d}$$

The dilatational wave solution for the m'th layer is given by

$$\nabla\cdot\mathbf{u} = \Delta_m = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Psi}{\partial x\partial z} + \frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Psi}{\partial x\partial z} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Then by virtue of the wave equation

$$\Delta_m = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2 \Phi}{\partial t^2}$$

or in the Fourier domain

$$\Delta_m = -\frac{p^2}{\alpha^2}\Phi$$

which rearranges to

$$-\frac{\alpha^2}{p^2}\Delta_m = \Phi \tag{3.5}$$

Similarly for rotational waves

$$\nabla \times \mathbf{u} = \omega_m = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Psi}{\partial x^2}\right) = -\frac{1}{2}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial \Psi}{\partial x^2}\right)$$

Once more using the wave equation

$$\omega_m = -\frac{1}{2}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial \Psi}{\partial x^2}\right) = -\frac{1}{2\beta^2}\frac{\partial^2 \Psi}{\partial t^2}$$

In the Fourier domain this can be rearranged into

$$\frac{2\beta^2}{p^2}\omega_m = \Psi \tag{3.6}$$

The dilatational waves can be written as a combination of up going and down going waves in the m'th layer by

$$\Delta_m = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = e^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}] \tag{3.7}$$

which is 2.1 of H53. Likewise, for the rotational waves

$$\omega_m = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = e^{i(pt-kx)}[\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}] \tag{3.8}$$

which is 2.2 of H53, and the Δ_m and ω_m terms are constants. The displacements corresponding to (3.7) and (3.8) are given by 2.3 and 2.4 of H53

$$u = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial z} = -\frac{\alpha_m^2}{p^2}\frac{\partial \Delta_m}{\partial x} - 2\frac{\beta_m^2}{p^2}\frac{\partial \omega_m}{\partial z} \tag{3.9}$$

$$w = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi}{\partial x} = -\frac{\alpha_m^2}{p^2}\frac{\partial \Delta_m}{\partial z} + 2\frac{\beta_m^2}{p^2}\frac{\partial \omega_m}{\partial x} \tag{3.10}$$

This shows the equivalence of Haskell's potential and the standard potentials if we recall (3.5) and (3.6)

Continuity of displacement across a boundary is satisfied if the velocity components are made continuous. Additionally, since the apparent velocity $c = \frac{p}{k}$ is the same in all layers we can introduce the dimensionless quantities

$$\frac{1}{c}\frac{\partial u}{\partial t} = \frac{\dot{u}}{c} \qquad \text{and} \qquad \frac{1}{c}\frac{\partial w}{\partial t} = \frac{\dot{w}}{c}$$

Then we express (3.9) by

$$\frac{\dot{u}}{c} = \frac{k}{p}(ip)\left[-\frac{\alpha_m^2}{p^2}\frac{\partial \Delta_m}{\partial x} - 2\frac{\beta_m^2}{p^2}\frac{\partial \omega_m}{\partial z}\right] = -\frac{ik}{p^2}\left[\alpha_m^2\frac{\partial \Delta_m}{\partial x} + 2\beta_m^2\frac{\partial \omega_m}{\partial z}\right]$$

where with (3.7)

$$\frac{\partial \Delta_m}{\partial x} = \frac{\partial}{\partial x}e^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}] = -ike^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}]$$

and with (3.8)

$$\frac{\partial \omega_m}{\partial z} = \frac{\partial}{\partial z}e^{i(pt-kx)}[\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z}] = ikr_{\beta_m}e^{i(pt-kx)}[-\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z}]$$

we can expand

$$\begin{aligned}
\frac{\dot{u}}{c} &= -\frac{ik}{p^2}\left[-ik\alpha_m^2\Delta_m - ikr_{\beta_m}2\beta_m^2(\omega_m^d e^{-ikr_\beta z} - \omega_m^u e^{ikr_\beta z})\right] e^{i(pt-kx)} \\
&= -\frac{k^2}{p^2}\left[\alpha_m^2\Delta_m + 2r_{\beta_m}\beta_m^2(\omega_m^d e^{-ikr_{\beta_m} z} - \omega_m^u e^{ikr_{\beta_m} z})\right] e^{i(pt-kx)} \\
&= -\left[\frac{\alpha_m^2}{c^2}[\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}] + 2\frac{r_{\beta_m}\beta_m^2}{c^2}[\omega_m^d e^{-ikr_{\beta_m} z} - \omega_m^u e^{ikr_{\beta_m} z}]\right] e^{i(pt-kx)} \\
&= -\left[\frac{\alpha_m^2}{c^2}[\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}] + r_{\beta_m}\gamma_m[\omega_m^d e^{-ikr_{\beta_m} z} - \omega_m^u e^{ikr_{\beta_m} z}]\right] e^{i(pt-kx)}
\end{aligned}$$

Noting Euler's identity

$$\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z} = \cos(kr_{\alpha_m} z)(\Delta_m^d + \Delta_m^u) - i\sin(kr_{\alpha_m} z)(\Delta_m^d - \Delta_m^u)$$

$$\omega_m^d e^{-ikr_{\beta_m} z} - \omega_m^u e^{ikr_{\beta_m} z} = \cos(kr_{\beta_m} z)(\omega_m^d - \omega_m^u) - i\sin(kr_{\beta_m} z)(\omega_m^d + \omega_m^u)$$

And finally obtain 2.7 of H53

$$\begin{aligned}
\frac{\dot{u}}{c} = \Big\{ &-\frac{\alpha_m^2}{c^2}[(\Delta_m^d + \Delta_m^u)\cos(kr_{\alpha_m} z) - i(\Delta_m^d - \Delta_m^u)\sin(kr_{\alpha_m} z)] \\
&-r_{\beta_m}\gamma_m[(\omega_m^d - \omega_m^u)\cos(kr_{\beta_m} z) - i(\omega_m^d + \omega_m^u)\sin(kr_{\beta_m} z)]\Big\} e^{i(pt-kx)} \qquad (3.11)
\end{aligned}$$

In a similar fashion

$$\frac{\dot{w}}{c} = \frac{k}{p}(ip)\left[-\frac{\alpha_m^2}{p^2}\frac{\partial\Delta_m}{\partial z} + 2\frac{\beta_m^2}{p^2}\frac{\partial\omega_m}{\partial x}\right] = -\frac{ik}{p^2}\left[\alpha_m^2\frac{\partial\Delta_m}{\partial z} - 2\beta_m^2\frac{\partial\omega_m}{\partial x}\right]$$

where with (3.7)

$$\frac{\partial\Delta_m}{\partial z} = ikr_{\alpha_m} e^{i(pt-kx)}[-\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}]$$

and with (3.8)

$$\frac{\partial\omega_m}{\partial x} = -ike^{ip(t-kx)}[\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}]$$

we can expand

$$\begin{aligned}
\frac{\dot{w}}{c} &= -\frac{ik}{p^2}\left[-r_{\alpha_m} ik\alpha_m^2(\Delta_m^d e^{-ikr_{\alpha_m}z} - \Delta_m^u e^{ikr_{\alpha_m}z}) - ik2\beta_m^2\omega_m\right] e^{ip(t-kx)} \\
&= -\frac{k^2}{p^2}[r_{\alpha_m}\alpha_m^2(\Delta_m^d e^{-ikr_{\alpha_m}z} - \Delta_m^u e^{ikr_{\alpha_m}z}) + 2\beta_m^2\omega_m]e^{ip(t-kx)} \\
&= -\frac{k^2}{p^2}[r_{\alpha_m}\alpha_m^2(\Delta_m^d e^{-ikr_{\alpha_m}z} - \Delta_m^u e^{ikr_{\alpha_m}z}) + 2\beta_m^2(\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z})] \\
&= -\left[r_{\alpha_m}\frac{\alpha_m^2}{c^2}(\Delta_m^d e^{-ikr_{\alpha_m}z} - \Delta_m^u e^{ikr_{\alpha_m}z}) + \gamma_m(\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^{ikr_{\beta_m}z})\right]
\end{aligned}$$

Then noting Euler's identities we obtain 2.8 of H53

$$\begin{aligned}
\frac{\dot{w}}{c} = \Big\{ &-r_{\alpha_m}\frac{\alpha_m^2}{c^2}[(\Delta_m^d - \Delta_m^u)\cos(kr_{\alpha_m}z) - i(\Delta_m^d + \Delta_m^u)\sin(kr_{\alpha_m}z)] \\
&+\gamma_m[(\omega_m^d + \omega_m^u)\cos(kr_{\beta_m}z) - i(\omega_m^d - \omega_m^u)\sin(kr_{\beta_m}z)]\Big\}e^{i(pt-kx)} \qquad (3.12)
\end{aligned}$$

3.1.3 The Stress Terms

Using the isotropic strain relationship we have

$$\sigma_{ij} = c_{ijkl}e_{kl} = (\lambda+\mu)\delta_{ij}e_{kk} + \mu e_{ij}$$

It follows that the normal stress is given by

$$\sigma = \sigma_{zz} = \lambda\frac{\partial u}{\partial x} + (\lambda+2\mu)\frac{\partial w}{\partial z}$$

where we have

$$\frac{\partial u}{\partial x} = -\frac{\alpha_m^2}{p^2}\frac{\partial^2\Delta_m}{\partial x^2} - 2\frac{\beta_m^2}{p^2}\frac{\partial^2\omega_m}{\partial x\partial z} \qquad \text{and} \qquad \frac{\partial w}{\partial z} = -\frac{\alpha_m^2}{p^2}\frac{\partial^2\Delta_m}{\partial z^2} + 2\frac{\beta_m^2}{p^2}\frac{\partial^2\omega_m}{\partial z\partial x}$$

then

$$\lambda_m\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = -\lambda_m\frac{\alpha_m^2}{p^2}\left[\frac{\partial^2\Delta_m}{\partial x^2} + \frac{\partial^2\Delta_m}{\partial z^2}\right] = \lambda_m\Delta_m$$

From this we write

$$\begin{aligned}\sigma &= \lambda_m \Delta_m + 2\mu_m \left[-\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z^2} + 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial z \partial x} \right] \\ &= (\rho_m \alpha_m^2 - 2\rho_m \beta_m^2)\Delta_m + 2\rho_m \beta_m^2 \left[-\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z^2} + 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial z \partial x} \right] \\ &= \rho_m \left(\alpha_m^2 \Delta_m - 2\beta_m^2 \Delta_m + 2\beta_m^2 \left[-\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z^2} + 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial z \partial x} \right] \right) \\ &= \rho_m \left(\alpha_m^2 \Delta_m + 2\beta_m^2 \left[-\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z^2} - \Delta_m + 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial z \partial x} \right] \right)\end{aligned}$$

and using

$$-\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial x^2} - \frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z^2} = \Delta_m$$

we simplify to 2.5 of H53

$$\sigma = \rho_m \left(\alpha_m^2 \Delta_m + 2\beta_m^2 \left[\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial x^2} + 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial z \partial x} \right] \right) \tag{3.13}$$

For shear stress we have

$$\tau = \sigma_{xz} = 2\mu e_{xz} = \rho_m \beta_m^2 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

where

$$\frac{\partial u}{\partial z} = -\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z \partial x} - 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial z^2} \qquad \text{and} \qquad \frac{\partial w}{\partial x} = -\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial z \partial x} + 2\frac{\beta_m^2}{p^2}\frac{\partial^2 \omega_m}{\partial x^z}$$

it follows that

$$\tau = \rho_m \beta_m^2 \left[-2\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial x \partial z} + 2\frac{\beta_m^2}{p^2}\left(\frac{\partial^2 \omega_m}{\partial x^2} - \frac{\partial^2 \omega_m}{\partial z^2} \right) \right]$$

and we arrive at 2.6 of H53

$$\tau = 2\rho_m \beta_m^2 \left[-\frac{\alpha_m^2}{p^2}\frac{\partial^2 \Delta_m}{\partial x \partial z} + \frac{\beta_m^2}{p^2}\left(\frac{\partial^2 \omega_m}{\partial x^2} - \frac{\partial^2 \omega_m}{\partial z^2} \right) \right] \tag{3.14}$$

We have already found

$$\frac{\partial \Delta_m}{\partial z} = ikr_{\alpha_m} e^{i(pt-kx)}[-\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}]$$

$$\frac{\partial \omega_m}{\partial x} = -ike^{ip(t-kx)}[\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}]$$

where the second derivatives are as follows

$$\frac{\partial^2 \omega_m}{\partial x^2} = -k^2 e^{ip(t-kx)}[\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}]$$

$$\frac{\partial^2 \Delta_m}{\partial z^2} = ikr_{\alpha_m} e^{i(pt-kx)}[ikr_{\alpha_m} \Delta_m^d e^{-ikr_{\alpha_m} z} + ikr_{\alpha_m} \Delta_m^u e^{ikr_{\alpha_m} z}]$$
$$= -k^2 r_{\alpha_m}^2 e^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}]$$

$$\frac{\partial^2 \omega_m}{\partial x \partial z} = k^2 r_{\beta_m} e^{i(pt-kx)}[-\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}]$$

$$\frac{\partial^2 \Delta_m}{\partial x \partial z} = k^2 r_{\alpha_m} e^{i(pt-kx)}[-\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}]$$

Now we find

$$\frac{\partial \Delta_m}{\partial x} = -ike^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}]$$

$$\frac{\partial \omega_m}{\partial z} = ikr_{\beta_m} e^{i(pt-kx)}[-\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}]$$

where the second derivatives are

$$\frac{\partial^2 \Delta_m}{\partial x^2} = -k^2 e^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m} z} + \Delta_m^u e^{ikr_{\alpha_m} z}]$$

$$\frac{\partial^2 \omega_m}{\partial z^2} = ikr_{\beta_m} e^{i(pt-kx)}[ikr_{\beta_m} \omega_m^d e^{-ikr_{\beta_m} z} + ikr_{\beta_m} \omega_m^u e^{ikr_{\beta_m} z}]$$
$$= -k^2 r_{\beta_m}^2 e^{i(pt-kx)}[\omega_m^d e^{-ikr_{\beta_m} z} + \omega_m^u e^{ikr_{\beta_m} z}]$$

Then we can plug in for (3.13)

$$
\begin{aligned}
\sigma &= \rho_m e^{i(pt-kx)}\alpha_m^2[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}] \\
&+ \rho_m e^{i(pt-kx)}2\beta_m^2\left(\frac{\alpha_m^2}{p^2}(-k^2)[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}]\right) \\
&+ \rho_m e^{i(pt-kx)}2\beta_m^2\left[2\frac{\beta_m^2}{p^2}k^2 r_{\beta_m}(-\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z})\right] \\
&= \rho_m e^{i(pt-kx)}\left(\left[\alpha_m^2 - 2\beta_m^2\frac{\alpha_m^2}{c^2}\right]\left[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}\right]\right) \\
&+ \rho_m e^{i(pt-kx)}\left(2\beta_m^2\left[2\frac{\beta_m^2}{c^2}r_{\beta_m}[-\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^d e^{ikr_{\beta_m}z}]\right]\right) \\
&= \rho_m e^{i(pt-kx)}\left(\alpha_m^2(1-\gamma_m)[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}] + 2\beta_m^2\gamma_m r_{\beta_m}[-\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z}]\right) \\
&= \rho_m e^{i(pt-kx)}\left(\alpha_m^2(1-\gamma_m)[\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}] - 2c^2\gamma_m^2 r_{\beta_m}[\omega_m^d e^{-ikr_{\beta_m}z} - \omega_m^u e^{ikr_{\beta_m}z}]\right)
\end{aligned}
$$

Finally using Euler identities we end up with 2.9 of H53

$$
\begin{aligned}
\sigma = -\rho_m\alpha_m^2(\gamma_m - 1)&\left[(\Delta_m^d + \Delta_m^u)\cos(kr_{\alpha_m}z) - i(\Delta_m^d - \Delta_m^u)\sin(kr_{\alpha_m}z)\right] \\
&- \rho_m c^2\gamma_m^2 r_{\beta_m}\left[(\omega_m^d - \omega_m^u)\cos(kr_{\beta_m}z) - i(\omega_m^d + \omega_m^u)\sin(kr_{\beta_m}z)\right] \quad (3.15)
\end{aligned}
$$

Likewise for shear stress we have

$$
\begin{aligned}
\tau &= 2\rho_m\beta_m^2\left(-\frac{\alpha_m^2}{p^2}k^2 r_{\alpha_m}e^{i(pt-kx)}[-\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}]\right) \\
&+ 2\rho_m\beta_m^2\Big\{\frac{\beta_m^2}{p^2}(-k^2)[\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z}] \\
&-\frac{\beta_m^2}{p^2}(-k^2)r_{\beta_m}^2[\omega_m^d e^{-ikr_{\beta_m}z} + \omega_m^u e^{ikr_{\beta_m}z}]\Big\}e^{i(pt-kx)} \\
&= 2\rho_m\frac{\beta_m^2}{c^2}\left(\alpha_m^2 r_{\alpha_m}e^{i(pt-kx)}[\Delta_m^d e^{-ikr_{\alpha_m}z} - \Delta_m^u e^{ikr_{\alpha_m}z}]\right) \\
&+ 2\rho_m\frac{\beta_m^2}{c^2}\left[\underbrace{\beta_m^2\left(r_{\beta_m}^2 - 1\right)}_{c^2 - 2\beta_m^2 = c^2(1-2\beta_m^2/c^2)}\quad[\omega_m^d e^{-ikr_{\alpha_m}z} + \omega_m^u e^{ikr_{\alpha_m}z}]\right]e^{i(pt-kx)} \\
&= \rho_m\alpha_m^2\gamma_m r_{\alpha_m}[-\Delta_m^d e^{-ikr_{\alpha_m}z} + \Delta_m^u e^{ikr_{\alpha_m}z}]e^{i(pt-kx)} \\
&- \rho_m c^2\gamma_m(\gamma_m - 1)[\omega_m^d e^{-ikr_{\alpha_m}z} + \omega_m^u e^{ikr_{\alpha_m}z}]e^{i(pt-kx)}
\end{aligned}
$$

Then with Euler identities we find 2.10 of H53

$$\tau = \Big\{ \rho_m \alpha_m^2 \gamma_m r_{\alpha_m} \left[(\Delta_m^d - \Delta_m^u) \cos(k r_{\alpha_m} z) - i(\Delta_m^d + \Delta_m^u) \sin(k r_{\alpha_m} z) \right] \\ - \rho_m c^2 \gamma_m (\gamma_m - 1) \left[(\omega_m^d + \omega_m^u) \cos(k r_{\beta_m} z) - i(\omega_m^d - \omega_m^u) \sin(k r_{\beta_m} z) \right] \Big\} e^{i(pt - kx)} \tag{3.16}$$

3.2 The Haskell Propagator Matrix

We now build a recurrence relation between layers based as proposed by H53, and consider the cases of unit input P, SV, and SH plane waves. At any interface between m and $m-1$ layers we have a relation for the stress and displacement at the interface as well and the solution to the wave equation in the m'th layer

$$\begin{bmatrix} \frac{\dot{u}_{m-1}}{c} \\ \frac{\dot{w}_{m-1}}{c} \\ \sigma_{m-1} \\ \tau_{m-1} \end{bmatrix} = E_m \begin{bmatrix} \Delta_m^d + \Delta_m^u \\ \Delta_m^d - \Delta_m^u \\ \omega_m^d - \omega_m^u \\ \omega_m^d + \omega_m^u \end{bmatrix} \tag{3.17}$$

The elements of E are determined by setting $z = 0$ in (3.11) to (3.16) or in H53 equation 2.10

$$E_m = \begin{bmatrix} -\frac{\alpha_m^2}{c^2} & 0 & -\gamma_m r_{\beta_m} & 0 \\ 0 & -\frac{\alpha_m^2}{c^2} r_{\alpha_m} & 0 & \gamma_m \\ -\rho_m \alpha_m^2 (\gamma_m - 1) & 0 & -\rho_m c^2 \gamma_m r_{\alpha_m} & 0 \\ 0 & \rho_m \alpha_m^2 \gamma_m r_{\alpha_m} & 0 & -\rho_m c^2 \gamma_m (\gamma_m - 1) \end{bmatrix} \tag{3.18}$$

Now consider the next interface down, or the m interface

$$\begin{bmatrix} \frac{\dot{u}_m}{c} \\ \frac{\dot{w}_m}{c} \\ \sigma_m \\ \tau_m \end{bmatrix} = D_m \begin{bmatrix} \Delta_m^d + \Delta_m^u \\ \Delta_m^d - \Delta_m^u \\ \omega_m^d - \omega_m^u \\ \omega_m^d + \omega_m^u \end{bmatrix} \tag{3.19}$$

where D_m is given by (3.11) to (3.16) evaluated at the layer thickness $z = d_m$, i.e.

$$D_m = \begin{bmatrix} -\frac{\alpha_m^2}{c^2}\cos P_m & i\frac{\alpha_m^2}{c^2}\sin P_m & -\gamma_m r_{\beta_m}\cos Q_m & i\gamma_m r_{\beta_m}\sin Q_m \\ i\frac{\alpha_m^2}{c^2} r_{\alpha_m}\sin P_m & -\frac{\alpha_m^2}{c^2} r_{\alpha_m}\cos P_m & -i\gamma_m \sin Q_m & \gamma_m \cos Q_m \\ -\rho_m\alpha_m^2(\gamma_m-1)\cos P_m & i\rho_m\alpha_m^2(\gamma_m-1)\sin P_m & -\rho_m c^2\gamma_m^2 r_{\beta_m}\cos Q_m & i\rho_m c^2\gamma_m^2 r_{\beta_m}\sin Q_m \\ -i\rho_m\alpha_m^2\gamma_m r_{\alpha_m}\sin P_m & \rho_m\alpha_m^2\gamma_m r_{\alpha_m}\cos P_m & i\rho_m c^2\gamma_m(\gamma_m-1)\sin Q_m & -\rho_m c^2\gamma_m(\gamma_m-1)\cos Q_m \end{bmatrix} \tag{3.20}$$

where $P_m = kr_{\alpha_m}d_m$ and $Q_m = kr_{\beta_m}d_m$. The constants $\Delta_m^u + \Delta_m^d$ etc. can be eliminated in (3.17) and (3.18) giving a linear relationship between the values $\dot{u}/c$, $\dot{w}/c$ σ, and τ at the top and bottom of the m'th layer which may be expressed by

$$\begin{bmatrix} \frac{\dot{u}_m}{c} \\ \frac{\dot{w}_m}{c} \\ \sigma_m \\ \tau_m \end{bmatrix} = D_m E_m^{-1} \begin{bmatrix} \frac{\dot{u}_{m-1}}{c} \\ \frac{\dot{w}_{m-1}}{c} \\ \sigma_{m-1} \\ \tau_{m-1} \end{bmatrix} = a_m \begin{bmatrix} \frac{\dot{u}_{m-1}}{c} \\ \frac{\dot{w}_{m-1}}{c} \\ \sigma_{m-1} \\ \tau_{m-1} \end{bmatrix} \tag{3.21}$$

where E_m^{-1} is the inverse of E_m given by

$$E_m^{-1} = \begin{bmatrix} -2\frac{\beta_m^2}{\alpha_m^2} & 0 & \frac{1}{\rho_m\alpha_m^2} & 0 \\ 0 & c^2\frac{(\gamma_m-1)}{\alpha_m^2 r_{\alpha_m}} & 0 & \frac{1}{\rho_m\alpha_m^2 r_{\alpha_m}} \\ \frac{(\gamma_m-1)}{\gamma_m r_{\beta_m}} & 0 & -\frac{1}{\rho_m c^2\gamma_m r_{\beta_m}} & 0 \\ 0 & 1 & 0 & \frac{1}{\rho_m c^2\gamma_m} \end{bmatrix} \tag{3.22}$$

From equations (3.20) and (3.22) the elements of the matrix product $a_m = D_m E_m^{-1}$ can be computed by

$$\begin{aligned} (a_m)_{11} =& \gamma_m \cos P_m - (\gamma_m - 1)\cos Q_m \\ (a_m)_{12} =& i[(\gamma_m - 1)r_{\alpha_m}^{-1}\sin P_m + \gamma_m r_{\beta_m}\sin Q_m \\ (a_m)_{13} =& -(\rho_m c^2)^{-1}(\cos P_m - \cos Q_m) \\ (a_m)_{14} =& i(\rho_m c^2)^{-1}(r_{\alpha_m}^{-1}\sin P_m + r_{\beta_m}\sin Q_m) \end{aligned}$$

$$
\begin{aligned}
(a_m)_{21} =& -i[\gamma_m r_{\alpha_m} \sin P_m + (\gamma_m - 1) r_{\beta_m}^{-1} \sin Q_m] \\
(a_m)_{22} =& -(\gamma_m - 1)\cos P_m + \gamma_m \cos Q_m \\
(a_m)_{23} =& i(\rho_m c^2)^{-1}(r_{\alpha_m} \sin P_m + r_{\beta_m}^{-1} \sin Q_m) \\
(a_m)_{24} =& (a_m)_{13}
\end{aligned}
$$

$$
\begin{aligned}
(a_m)_{31} =& \rho_m c^2 \gamma_m (\gamma_m - 1)(\cos P_m - \cos Q_m) \\
(a_m)_{32} =& i\rho_m c^2 [(\gamma_m - 1)^2 r_{\alpha_m}^{-1} \sin P_m + \gamma_m^2 r_{\beta_m} \sin Q_m] \\
(a_m)_{33} =& (a_m)_{22} \\
(a_m)_{34} =& (a_m)_{12}
\end{aligned}
$$

$$
\begin{aligned}
(a_m)_{41} =& i\rho_m c^2 [\gamma_m^2 r_{\alpha_m} \sin P_m + (\gamma_m - 1)^2 r_{\beta_m}^{-1} \sin Q_m] \\
(a_m)_{42} =& (a_m)_{31} \\
(a_m)_{43} =& (a_m)_{21} \\
(a_m)_{44} =& (a_m)_{11}
\end{aligned}
$$

Now the boundary conditions require the values of $\frac{\dot{u}}{c}$, $\frac{\dot{w}}{c}$, σ, and τ computed at the top of the m'th layer be the same as the values computed at the bottom of the $(m-1)$'th layer. Then we write 2.17 of H53 as

$$
\begin{bmatrix} \frac{\dot{u}_m}{c} \\ \frac{\dot{w}_m}{c} \\ \sigma_m \\ \tau_m \end{bmatrix} = a_m a_{m-1} \begin{bmatrix} \frac{\dot{u}_{m-2}}{c} \\ \frac{\dot{w}_{m-2}}{c} \\ \sigma_{m-2} \\ \tau_{m-2} \end{bmatrix} \tag{3.23}
$$

Repeated calculation of (3.23) yields 2.18 of H53

$$
\begin{bmatrix} \frac{\dot{u}_{n-1}}{c} \\ \frac{\dot{w}_{n-1}}{c} \\ \sigma_{n-1} \\ \tau_{n-1} \end{bmatrix} = a_{n-1} a_{n-2} \cdots a_1 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ \sigma_0 \\ \tau_0 \end{bmatrix} \tag{3.24}
$$

Then application of the inverse yields 2.19 of H53

$$\begin{bmatrix} \Delta_n^d + \Delta_n^u \\ \Delta_n^d - \Delta_n^u \\ \omega_n^d - \omega_n^u \\ \omega_n^d + \omega_n^u \end{bmatrix} = E_n^{-1} a_{n-1} a_{n-2} \cdots a_1 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ \sigma_0 \\ \tau_0 \end{bmatrix} \tag{3.25}$$

Of particular interest is the case when the stresses vanish at the free surface and there are no sources at infinity, so that $\Delta_n^u = \omega_n^u = 0$. We can write (3.25)

$$\begin{bmatrix} \Delta_n^d \\ \Delta_n^d \\ \omega_n^d \\ \omega_n^d \end{bmatrix} = E_n^{-1} a_{n-1} a_{n-2} \cdots a_1 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = J \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} \tag{3.26}$$

or explicitly as

$$\Delta_n^d = J_{11} \frac{\dot{u}_0}{c} + J_{12} \frac{\dot{w}_0}{c} \tag{3.27a}$$

$$\Delta_n^d = J_{21} \frac{\dot{u}_0}{c} + J_{22} \frac{\dot{w}_0}{c} \tag{3.27b}$$

$$\omega_n^d = J_{31} \frac{\dot{u}_0}{c} + J_{32} \frac{\dot{w}_0}{c} \tag{3.27c}$$

$$\omega_n^d = J_{41} \frac{\dot{u}_0}{c} + J_{42} \frac{\dot{w}_0}{c} \tag{3.27d}$$

By eliminating Δ_n^d and ω_n^d we have

$$\frac{\dot{u}_0}{\dot{w}_0} = \frac{J_{22} - J_{12}}{J_{11} - J_{21}} = \frac{J_{42} - J_{32}}{J_{31} - J_{41}} \tag{3.28}$$

As noted by H53 since J is a function of c and k, (3.28) provides an implicit relationship between c and k, which is the phase velocity dispersion function.

Of particular interest to us, and not so much Haskell, is the displacement at any depth within the medium. Suppose this depth is a distance h below interface $m-1$. Then

$$\begin{bmatrix} \frac{\dot{u}_h}{c} \\ \frac{\dot{w}_h}{c} \\ \sigma_h \\ \tau_h \end{bmatrix} = D_m(h)E_m^{-1} \begin{bmatrix} \frac{\dot{u}_{m-1}}{c} \\ \frac{\dot{w}_{m-1}}{c} \\ \sigma_{m-1} \\ \tau_{m-1} \end{bmatrix} = a(h) \begin{bmatrix} \frac{\dot{u}_{m-1}}{c} \\ \frac{\dot{w}_{m-1}}{c} \\ \sigma_{m-1} \\ \tau_{m-1} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\dot{u}_h}{c} \\ \frac{\dot{w}_h}{c} \\ \sigma_h \\ \tau_h \end{bmatrix} = a_h a_{m-2} \cdots a_1 a_0 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ \sigma_0 \\ \tau_0 \end{bmatrix} = M \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \\ M_{41} & M_{42} \end{bmatrix}$$

Then using the structure of M, and realizing only M_{11}, M_{12}, M_{21} and M_{22} will be active by consequence of matrix multiplication gives

$$\begin{bmatrix} \frac{\dot{u}_h}{c} \\ \frac{\dot{w}_h}{c} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \end{bmatrix}$$

where

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & iC_{12} \\ iC_{21} & C_{22} \end{bmatrix}$$

then

$$\frac{\dot{u}_h}{c} = C_{11}\frac{\dot{u}_0}{c} + iC_{12}\frac{\dot{w}_0}{c} = C_{11}\Re(U) - C_{12}\Im(W) + i[C_{11}\Im(U) + C_{12}\Re(W)]$$
$$\frac{\dot{w}_h}{c} = iC_{21}\frac{\dot{u}_0}{c} + C_{22}\frac{\dot{w}_0}{c} = -C_{21}\Im(U) + C_{22}\Re(W) + i[C_{21}\Re(U) + C_{22}\Im(W)]$$

Hence, after calculating the repsonse at the surface due to the input of our choice, we then back out the response anywhere else.

3.2.1 The Incident P Wave

For the case of an incident P wave of unit amplitude, and no up going S wave we have $\Delta_n^u = 1$ and $\omega_n^u = 0$, or the matrix analog of (3.25) in terms of reflectivity

coefficients

$$\begin{bmatrix} \Delta_n^d + 1 \\ \Delta_n^d - 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \\ J_{41} & J_{42} \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ \sigma_0 \\ \tau_0 \end{bmatrix}$$

or expanded as

$$1 = J_{11}\frac{\dot{u}_0}{c} + J_{12}\frac{\dot{w}_0}{c} - \Delta_n^d$$

$$-1 = J_{21}\frac{\dot{u}_0}{c} + J_{22}\frac{\dot{w}_0}{c} - \Delta_n^d$$

$$0 = J_{31}\frac{\dot{u}_0}{c} + J_{32}\frac{\dot{w}_0}{c} - \omega_n^d$$

$$0 = J_{41}\frac{\dot{u}_0}{c} + J_{42}\frac{\dot{w}_0}{c} - \omega_n^d$$

which is 4 equations with 4 unknowns. Rearranging we find

$$\begin{bmatrix} -1 & 0 & J_{11} & J_{12} \\ -1 & 0 & J_{21} & J_{22} \\ 0 & -1 & J_{31} & J_{32} \\ 0 & -1 & J_{41} & J_{42} \end{bmatrix} \begin{bmatrix} \Delta_n^d \\ \omega_n^d \\ \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Now our goal is to invert and solve for the Δ_n^d, ω_n^d, $\frac{\dot{u}}{c}$, and $\frac{\dot{w}}{c}$, where we will call R our inverse matrix. Finding the determinant of a 4×4 system we use the transpose

$$\begin{vmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ J_{11} & J_{21} & J_{31} & J_{41} \\ J_{12} & J_{22} & J_{32} & J_{42} \end{vmatrix} = - \begin{vmatrix} 0 & -1 & -1 \\ J_{21} & J_{31} & J_{41} \\ J_{22} & J_{32} & J_{42} \end{vmatrix} + \begin{vmatrix} 0 & -1 & -1 \\ J_{11} & J_{31} & J_{41} \\ J_{12} & J_{32} & J_{42} \end{vmatrix}$$

which expands to

$$= -(J_{21}J_{42} - J_{41}J_{22}) + (J_{21}J_{32} - J_{31}J_{22}) + (J_{11}J_{42} - J_{41}J_{12}) - (J_{11}J_{32} - J_{31}J_{12})$$
$$= -J_{21}(J_{42} - J_{32}) + J_{22}(J_{41} - J_{31}) + J_{11}(J_{42} - J_{32}) - J_{12}(J_{41} - J_{31})$$

and our corrected determinant from equation 7 of H62.

$$D = (J_{11} - J_{21})(J_{42} - J_{32}) - (J_{12} - J_{22})(J_{41} - J_{31}) \tag{3.30}$$

We can also find the elements of the inverse by taking the determinants of the 3×3 sub-matrices

$$R_{11} = \begin{vmatrix} 0 & J_{21} & J_{22} \\ -1 & J_{31} & J_{32} \\ -1 & J_{41} & J_{42} \end{vmatrix} = J_{21}(J_{42} - J_{32}) + J_{22}(J_{31} - J_{41})$$

$$R_{21} = \begin{vmatrix} -1 & J_{21} & J_{22} \\ 0 & J_{31} & J_{32} \\ 0 & J_{41} & J_{42} \end{vmatrix} = J_{32}J_{41} - J_{31}J_{42}$$

$$R_{31} = \begin{vmatrix} -1 & 0 & J_{22} \\ 0 & -1 & J_{32} \\ 0 & -1 & J_{42} \end{vmatrix} = J_{42} - J_{32} \qquad R_{41} = \begin{vmatrix} -1 & 0 & J_{21} \\ 0 & -1 & J_{31} \\ 0 & -1 & J_{41} \end{vmatrix} = J_{41} - J_{31}$$

$$R_{12} = \begin{vmatrix} 0 & J_{11} & J_{12} \\ -1 & J_{31} & J_{32} \\ -1 & J_{41} & J_{42} \end{vmatrix} = J_{11}(J_{42} - J_{32}) + J_{12}(J_{31} - J_{41})$$

$$R_{22} = \begin{vmatrix} -1 & J_{11} & J_{12} \\ 0 & J_{31} & J_{32} \\ 0 & J_{41} & J_{42} \end{vmatrix} = J_{32}J_{41} - J_{31}J_{42}$$

$$R_{32} = \begin{vmatrix} -1 & 0 & J_{12} \\ 0 & -1 & J_{32} \\ 0 & -1 & J_{42} \end{vmatrix} = J_{42} - J_{32} \qquad R_{42} = \begin{vmatrix} -1 & 0 & J_{11} \\ 0 & -1 & J_{31} \\ 0 & -1 & J_{41} \end{vmatrix} = J_{41} - J_{31}$$

$$R_{13} = \begin{vmatrix} 0 & J_{11} & J_{12} \\ 0 & J_{21} & J_{22} \\ -1 & J_{41} & J_{42} \end{vmatrix} = J_{12}J_{21} - J_{11}J_{22}$$

$$R_{23} = \begin{vmatrix} -1 & J_{11} & J_{12} \\ -1 & J_{21} & J_{22} \\ 0 & J_{41} & J_{42} \end{vmatrix} = J_{41}(J_{22} - J_{12}) + J_{42}(J_{11} - J_{21})$$

$$R_{33} = \begin{vmatrix} -1 & 0 & J_{12} \\ -1 & 0 & J_{22} \\ 0 & -1 & J_{42} \end{vmatrix} = J_{12} - J_{22} \qquad R_{43} = \begin{vmatrix} -1 & 0 & J_{11} \\ -1 & 0 & J_{21} \\ 0 & -1 & J_{41} \end{vmatrix} = J_{11} - J_{21}$$

$$R_{14} = \begin{vmatrix} 0 & J_{11} & J_{12} \\ 0 & J_{21} & J_{22} \\ -1 & J_{31} & J_{32} \end{vmatrix} = J_{12}J_{21} - J_{11}J_{22}$$

$$R_{24} = \begin{vmatrix} -1 & J_{11} & J_{12} \\ -1 & J_{21} & J_{22} \\ 0 & J_{31} & J_{32} \end{vmatrix} = J_{31}(J_{22} - J_{12}) + J_{32}(J_{11} - J_{21})$$

$$R_{34} = \begin{vmatrix} -1 & 0 & J_{12} \\ -1 & 0 & J_{22} \\ 0 & -1 & J_{32} \end{vmatrix} = J_{12} - J_{22} \qquad R_{43} = \begin{vmatrix} -1 & 0 & J_{11} \\ -1 & 0 & J_{21} \\ 0 & -1 & J_{31} \end{vmatrix} = J_{11} - J_{21}$$

From this we could reassemble R in it's entirety, however,since we are effectively only multiplying by the vector $[1\,,-1]^T$, we can truncate R to

$$R = \begin{bmatrix} J_{21}(J_{42} - J_{32}) + J_{22}(J_{31} - J_{41}) & -J_{11}(J_{42} - J_{32}) - J_{12}(J_{31} - J_{41}) \\ -J_{32}J_{41} + J_{31}J_{42} & J_{32}J_{41} - J_{31}J_{42} \\ J_{42} - J_{32} & -J_{42} + J_{32} \\ -J_{41} + J_{31} & J_{41} - J_{31} \end{bmatrix}$$

Finally doing the multiplication yields

$$\Delta_n^d = \frac{J_{21}(J_{42} - J_{32}) + J_{22}(J_{31} - J_{41}) + J_{11}(J_{42} - J_{32}) + J_{12}(J_{31} - J_{41})}{D}$$

$$\Delta_n^d = \frac{(J_{11} + J_{21})(J_{42} - J_{32}) - (J_{12} + J_{22})(J_{41} - J_{31})}{D} \tag{3.31}$$

$$\omega_n^d = \frac{-J_{32}J_{41} + J_{31}J_{42} - J_{32}J_{41} + J_{31}J_{42}}{D} = 2\frac{(J_{31}J_{42} - J_{32}J_{41})}{D} \tag{3.32}$$

$$\frac{\dot{u}}{c} = \frac{J_{42} - J_{32} + J_{42} - J_{32}}{D} = 2\frac{(J_{42} - J_{32})}{D} \tag{3.33}$$

$$\frac{\dot{w}}{c} = \frac{-J_{41} + J_{31} - J_{41} + J_{31}}{D} = 2\frac{(J_{31} - J_{41})}{D} \tag{3.34}$$

These are the corrected quantities for H62.

We can rewrite D a bit

$$D = -(J_{11}J_{32} - J_{21}J_{32}) + (J_{11}J_{42} - J_{21}J_{42}) + (J_{12}J_{31} - J_{22}J_{31}) - (J_{12}J_{41} - J_{22}J_{41})$$

then

$$\Re(D) = J_{11}J_{42} - J_{21}J_{32} - J_{22}J_{31} + J_{12}J_{41}$$

$$\Im(D) = -J_{11}J_{32} - J_{21}J_{42} + J_{12}J_{31} + J_{22}J_{41}$$

Note that

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \\ J_{41} & J_{42} \end{bmatrix} = \begin{bmatrix} \Re & \Im \\ \Im & \Re \\ \Re & \Im \\ \Im & \Re \end{bmatrix} = \begin{bmatrix} C_{11} & iC_{12} \\ iC_{21} & C_{22} \\ C_{31} & iC_{32} \\ iC_{41} & C_{42} \end{bmatrix}$$

$$D = -(iC_{11}C_{32}+C_{21}C_{32})+(C_{11}C_{42}-iC_{21}C_{42})+(iC_{12}C_{31}-C_{22}C_{31})+(C_{12}C_{41}+iC_{22}C_{41})$$

$$\Re(D) = C_{11}C_{42} - C_{21}C_{32} - C_{22}C_{31} + C_{12}C_{41}$$

$$\Im(D) = -C_{11}C_{32} - C_{21}C_{42} + C_{12}C_{31} + C_{22}C_{41}$$

From (3.17) and (3.18) the values of $\frac{\dot{u}}{c}$ and $\frac{\dot{w}}{c}$ at the top of the n'th layer are given

by

$$\frac{\dot{u}_{n-1}}{c} = -\frac{\alpha_n^2}{c^2}(\Delta_n^d + \Delta_n^u) - \gamma_n r_{\beta_n}(\omega_n^d - \omega_n^u) \tag{3.35}$$

$$\frac{\dot{w}_{n-1}}{c} = -\frac{\alpha_n^2}{c^2} r_{\alpha_n}(\Delta_n^d - \Delta_n^u) + \gamma_n(\omega_n^d + \omega_n^u) \tag{3.36}$$

To exhibit combined effect of crustal layering and the free surface we can normalize (3.33) and (3.34) by the coefficients of Δ_n^u in (3.35) and Δ_n^u in (3.36), thus the normalized surface amplitudes are

$$U_p = 2\frac{c^2}{\alpha_n^2}\frac{(J_{32} - J_{42})}{D} \tag{3.37}$$

$$W_p = 2\frac{c^2}{\alpha_n^2 r_{\alpha_n}}\frac{(J_{31} - J_{41})}{D} \tag{3.38}$$

H62 notes these as the transfer functions relating the given input component of motion to the output free surface component of motion.

If we seek to normalize the surface amplitudes to unit total amplitude in the incident wave, then we should multiply the right hand side of (3.37) by $\frac{\alpha_n}{c} = \sin(i_{pn})$ and the right hand side of (3.38) by $\frac{\alpha_n r_{\alpha_n}}{c} = \cos(i_{pn})$

$$\tilde{U}_p = 2\frac{c}{\alpha_n}\frac{(J_{32} - J_{42})}{D}$$

$$\tilde{W}_p = 2\frac{c}{\alpha_n}\frac{(J_{31} - J_{41})}{D}$$

which is the response to the total excitation for $\Delta_n^d = 1$.

For implementation reasons, we desire normalized forms for the real and imaginary parts of U_p and W_p. Note that if $D = x + iy$, then

$$\frac{1}{D} = \frac{D^*}{DD^*} = \frac{x - iy}{DD^*}$$

$$U_p = \frac{2c}{\alpha_n}\frac{(J_{32} - J_{42})}{D} = \frac{2c}{\alpha_n}\frac{D^*(J_{32} - J_{42})}{DD^*} = \frac{2c}{\alpha_n|D|^2}(x - iy)(J_{32} - J_{42}) = t(x - iy)(iC_{32} - C_{42})$$

$$(x - iy)(iC_{32} - C_{42}) = ixC_{32} - xC_{42} + yC_{32} + iyC_{42}$$

$$\Re(U_p) = -t(xC_{42} - yC_{32}) \tag{3.39a}$$

$$\Im(U_p) = t(yC_{42} + xC_{32}) \tag{3.39b}$$

Expressions for the normalized real and imaginary parts of W_p displacements can be found by

$$(x - iy)(J_{31} - J_{41}) = xJ_{31} - iyJ_{31} - xJ_{41} + iyJ_{41} = xC_{31} - iyC_{31} - ixC_{41} - yC_{41}$$

$$\Re(W_p) = t(xC_{31} - yC_{41}) \tag{3.40a}$$

$$\Im(W_p) = -t(xC_{41} + yC_{31}) \tag{3.40b}$$

Finally we can determine the real and imaginary parts of the ratio $\frac{U_p}{W_p}$

$$\frac{U_p}{W_p} = \frac{U_p W_p^*}{W_p W_p^*} = \frac{\Re(U_p)\Re(W_p) + \Im(U_p)\Im(W_p) + i[\Im(U_p)\Re(W_p) - \Re(U_p)\Im(W_p)]}{W_p W_p^*}$$

If we set $\Delta = 1$ in (3.35) and (3.36) then we have velocities scaled by a medium constant. So our input is the velocity impulse (which has a flat spectrum) and the quantity we recover are the dimensionless velocities $U = \frac{\dot{u}}{c}$ and $W = \frac{\dot{w}}{c}$. To recover the associated displacement we integrate over time

$$u = c \int U\, dt = \frac{c}{ip} U = -i\frac{c}{p} U \qquad \text{and} \qquad w = -i\frac{c}{p} W$$

$$(u_r, u_i) = -i\frac{c}{p}(U_r, U_i) = \frac{c}{p}(U_i, -U_r)$$

where we take u and w to be zero at zero frequency, $p = 0$. In general the displacement associated with a velocity impulse will not be of much use.

If we want the response to a displacement impulse δ, we need to scale Δ appropriately. The velocity spectrum Δ will be related to an equivalent displacement spectrum δ by $\Delta = ip\delta$. For a flat spectrum associated with displacement, $\delta = 1$, it follows that $\Delta = ip$. Then the velocity response to a displacement impulse is just

$ipU = U_d$, with associated displacement

$$u = c \int U_d dt = \frac{c}{ip} ipU = cU$$

or, in other words, the displacement response to a displacement impulse is the same as the velocity response to a velocity impulse.

For some applications we need to know the displacement response to an acceleration impulse, a. Here, the time integration means the velocity spectrum, Δ, will be related to an equivalent acceleration spectrum, a, by $\Delta = \frac{a}{ip}$. It follows, for a flat acceleration spectrum, $a = 1$, that $\Delta = \frac{1}{ip}$. The velocity response to an acceleration impulse is thus $\frac{U}{ip} = U_a$. The associated displacement is then

$$u = c \int U_a dt = \frac{c}{ip} \frac{U}{ip} = -\frac{c}{p^2} U$$

In the general case of Green's functions for unit forces, it would seem that we want this last form. For example, in the wave tomography software, the source term units are force per volume, therefore we would like the Green's functions corresponding to a displacement response from an acceleration impulse.

3.2.2 The Incident SV Wave

For the case of incident SV waves in the n'th layer, we set $\Delta_n^u = 0$ and $\omega_n^u = 1$ and once more consider the incident wave to have unit amplitude. This reduces (3.25) to

$$\begin{bmatrix} \Delta_n^d \\ \Delta_n^d \\ \omega_n^d - 1 \\ \omega_n^d + 1 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \\ J_{41} & J_{42} \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \end{bmatrix}$$

which can be written as

$$0 = J_{11} \frac{\dot{u}_0}{c} + J_{12} \frac{\dot{w}_0}{c} - \Delta_n^d$$

$$0 = J_{21}\frac{\dot{u}_0}{c} + J_{22}\frac{\dot{w}_0}{c} - \Delta_n^d$$

$$-1 = J_{31}\frac{\dot{u}_0}{c} + J_{32}\frac{\dot{w}_0}{c} - \omega_n^d$$

$$1 = J_{41}\frac{\dot{u}_0}{c} + J_{42}\frac{\dot{w}_0}{c} - \omega_n^d$$

then rearranging as 4 equations and 4 unknowns gives

$$\begin{bmatrix} -1 & 0 & J_{11} & J_{12} \\ -1 & 0 & J_{21} & J_{22} \\ 0 & -1 & J_{31} & J_{32} \\ 0 & -1 & J_{41} & J_{42} \end{bmatrix} \begin{bmatrix} \Delta_n^d \\ \omega_n^d \\ \frac{\dot{u}}{c} \\ \frac{\dot{w}}{c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The work done for our previous inverse is still applicable however we now realize that the final two columns are active, thus

$$\begin{bmatrix} \Delta_n^d \\ \omega_n^d \\ \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} J_{12}J_{21} - J_{11}J_{22} & -J_{12}J_{21} + J_{11}J_{22} \\ -J_{42}(J_{11} - J_{21}) - J_{41}(J_{22} - J_{12}) & J_{31}(J_{22} - J_{12}) + J_{32}(J_{11} - J_{21}) \\ J_{12} - J_{22} & -J_{12} + J_{22} \\ -J_{11} + J_{21} & J_{11} - J_{21} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This yields

$$\Delta_n^d = 2\frac{(J_{11}J_{22} - J_{12}J_{21})}{D} \tag{3.42}$$

$$\omega_n^d = \frac{J_{42}(J_{11} - J_{21}) + J_{41}(J_{22} - J_{12}) + J_{31}(J_{22} - J_{12}) + J_{32}(J_{11} - J_{21})}{D}$$

$$\omega_n^d = \frac{(J_{22} - J_{12})(J_{31} + J_{41}) - (J_{21} - J_{11})(J_{32} + J_{42})}{D} \tag{3.43}$$

$$\frac{\dot{u}_0}{c} = 2\frac{(J_{22} - J_{12})}{D} \tag{3.44}$$

$$\frac{\dot{w}_0}{c} = 2\frac{(J_{11} - J_{21})}{D} \tag{3.45}$$

As before we can find the normalized surface amplitudes by dividing (3.44) and (3.45) by the ω_n^u coefficients in (3.35) and (3.36), respectively,

$$U_s = \frac{2}{\gamma_n r_{\beta_n}} \frac{(J_{22} - J_{12})}{D} \tag{3.46}$$

$$W_s = \frac{2}{\gamma_n} \frac{(J_{11} - J_{21})}{D} \tag{3.47}$$

To normalize with respect to the total amplitude of the incident wave we use

$$U_s = \frac{2}{\gamma_n r_{\beta_n}} \frac{(J_{22} - J_{12})}{D} \frac{\beta_n r_{\beta_n}}{c} = \frac{c(C_{22} - iC_{12})}{\beta_n D}$$

$$W_s = \frac{2}{\gamma_n} \frac{(J_{11} - J_{21})}{D} \frac{\beta_n}{c} = \frac{c(C_{11} - iC_{21})}{\beta_n D}$$

If the wave is traveling horizontally, then it is pure SV. H62 notes we could resolve it into an SV and SH using the angle of polarization ϵ of the incident wave

$$\tan \epsilon = \frac{A_{SH}}{A_{SV}} \tag{3.48}$$

Then if we let V_s be the normalized surface amplitude for SH waves, $\frac{v_0}{v_n^d}$, as defined by equation 5 of H60, the surface amplitudes are given by

$$u_0 = A_{SV} U_s \cos i_{sn} \tag{3.49a}$$

$$v_0 = A_{SV} V_s \tan \epsilon \tag{3.49b}$$

$$w_0 = A_{SV} W_s \sin i_{sn} \tag{3.49c}$$

but that is not really needed here. To obtain the real and imaginary parts we proceed as before

$$U_s = \frac{c}{\beta_n} \frac{(C_{22} - iC_{22})}{D} = \frac{c}{\beta_n} \frac{D^*(C_{22} - iC_{22})}{DD^*} = \frac{c}{\beta_n |D|^2} (x - iy)(C_{22} - iC_{12})$$

with the real and imaginary parts given by

$$\Re(U_s) = \frac{c}{\beta_n |D|^2} (xC_{22} - yC_{12})$$

$$\Im(U_s) = -\frac{c}{\beta_n |D|^2}(xC_{12} + yC_{22})$$

Then for W_s

$$W_s = \frac{c(C_{11} - iC_{21})}{\beta_n D} = \frac{c}{\beta_n}\frac{D^*(C_{11} - iC_{21})}{DD^*} = \frac{c}{\beta_n |D|^2}(x - iy)(C_{11} - iC_{21})$$

which gives

$$\Re(W_s) = \frac{c}{\beta_n |D|^2}(xC_{11} - yC_{21})$$

$$\Im(W_s) = -\frac{c}{\beta_n |D|^2}(xC_{21} + yC_{21})$$

3.2.3 The Incident SH Wave

Following from H60 we now consider the case of the incident SH wave. The plane wave solution for the homogenous elastic equation of motion for the SH case is given by H53 (9.1)

$$v = e^{i(pt - k_x x)}[v^d e^{-ik_x r_\beta z} + v^u e^{ik_x r_\beta z}] \tag{3.50}$$

with corresponding transverse shearing stress

$$Y_s = \mu\frac{\partial v}{\partial z} = ik_x \mu r_\beta e^{i(pt - k_x x)}[-v^d e^{-ik_x r_\beta z} + v^u e^{ik_x r_\beta z}] \tag{3.51}$$

If we differentiate (3.50) with respect to time we find

$$\dot{v} = ipe^{ipt}e^{-ik_x x}[v^d e^{-ik_x r_\beta z} + v^u e^{ik_x r_\beta z}] = ik_x c e^{ipt}e^{-ik_x x}[v^d e^{-ik_x r_\beta z} + v^u e^{ik_x r_\beta z}]$$

If we set $z = 0$ at the $(m-1)$'th interface we then have

$$\left(\frac{\dot{v}}{c}\right)_{m-1} = ik_(v_m^d + v_m^u)e^{i(pt - k_x x)} \tag{3.52a}$$

$$Y_{s_{m-1}} = ik_x \mu_m r_{\beta_m}(v_m^u - v_m^d)e^{i(pt - k_x x)} \tag{3.52b}$$

which can be rewritten as

$$(v_m^d + v_m^u) = -\frac{i}{k_x}\left(\frac{\dot{v}}{c}\right)_{m-1} e^{-ipt}e^{ik_x x} \qquad (v_m^u - v_m^d) = -\frac{i}{k_x \mu_m r_{\beta_m}} Y_{s_{m-1}} e^{-ipt}e^{ik_x x}$$

Then at the m'th interface, $z = d$, we find

$$\left(\frac{\dot{v}}{c}\right)_m = ik_x[v^d e^{-ik_x r_{\beta_m} z} + v^u e^{ik_x r_{\beta_m} z}]e^{ipt}e^{-ik_x x}$$

Defining $Q_m \equiv k_x r_{\beta_m} d$

$$\begin{aligned}\left(\frac{\dot{v}}{c}\right)_m &= ik_x[v_m^d e^{-iQ_m} + v_m^u e^{iQ_m}]e^{ipt}e^{-ik_x x} \\ &= ik_x[v_m^d(\cos Q_m - i\sin Q_m) + v_m^u(\cos Q_m + i\sin Q_m)]e^{ipt}e^{-ik_x x} \\ &= ik_x[\cos Q_m(v_m^d + v_m^u) + i\sin Q_m(v_m^u - v_m^d)]e^{ipt}e^{-ik_x x}\end{aligned}$$

$$\begin{aligned}Y_{s_{m-1}} &= ik_x\mu_m r_{\beta_m}[-v_m^d e^{-ik_x r_{\beta_m} z} + v_m^u e^{ik_x r_{\beta_m} z}]e^{ipt}e^{-ik_x x} \\ &= ik_x\mu_m r_{\beta_m}[-v_m^d(\cos Q_m - i\sin Q_m) + v_m^u(\cos Q_m + i\sin Q_m)]e^{ipt}e^{-ik_x x}\end{aligned}$$

The above then becomes

$$\left(\frac{\dot{v}}{c}\right)_m = [ik_x\cos Q_m(v_m^d + v_m^u) - k_x\sin Q_m(v_m^u - v_m^d)]e^{ipt}e^{-ik_x x} \tag{3.53a}$$

$$Y_{s_{m-1}} = k_x\mu_m r_{\beta_m}[i\cos Q_m(v_m^u - v_m^d) - \sin Q_m(v_m^d + v_m^u)]e^{ipt}e^{-ik_x x} \tag{3.53b}$$

Substituting our rearranged expressions for $v_m^d + v_m^u$ and $v_m^u - v_m^d$ gives

$$\left(\frac{\dot{v}}{c}\right)_m = \left(\frac{\dot{v}}{c}\right)_{m-1}\cos Q_m + \frac{i}{\mu_m r_{\beta_m}} Y_{s_{m-1}}\sin Q_m \tag{3.54a}$$

$$Y_{s_m} = Y_{s_{m-1}}\cos Q_m + i\left(\frac{\dot{v}}{c}\right)_m \mu_m r_{\beta_m}\sin Q_m \tag{3.54b}$$

Hence

$$\begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_m \\ Y_{s_m} \end{bmatrix} = \begin{bmatrix} \cos Q_m & \frac{i}{\mu_m r_{\beta_m}} \sin Q_m \\ i\mu_m r_{\beta_m} \sin Q_m & \cos Q_m \end{bmatrix} \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_{m-1} \\ Y_{s_{m-1}} \end{bmatrix} = a_m \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_{m-1} \\ Y_{s_{m-1}} \end{bmatrix}$$

where

$$a_m = \begin{bmatrix} \cos Q_m & \frac{i}{\mu_m r_{\beta_m}} \sin Q_m \\ i\mu_m r_{\beta_m} \sin Q_m & \cos Q_m \end{bmatrix} \tag{3.55}$$

Applying the recurrence relation and a free surface it follows that

$$\begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_m \\ Y_{s_m} \end{bmatrix} = a_m a_{m-1} \cdots a_0 \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_0 \\ 0 \end{bmatrix} = A \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_0 \\ 0 \end{bmatrix}$$

At the bottom of the $(n-1)$ interface we see that

$$\begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_{n-1} \\ Y_{s_{n-1}} \end{bmatrix} = a_{n-1} a_{n-2} \cdots a_0 \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_0 \\ 0 \end{bmatrix} = A \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_0 \\ 0 \end{bmatrix}$$

Note

$$(v^d + v^u)_n = -\frac{i}{k_x} \left(\frac{\dot{v}}{c}\right)_{n-1} e^{-ipt} e^{ik_x x} = -\frac{i}{k_x} e^{-ipt} e^{ik_x x} [A_{11} \left(\frac{\dot{v}}{c}\right)_0 + A_{12} Y_{s_0}]$$

$$(v^u - v^d)_n = -\frac{i}{k_x \mu_n r_{\beta_n}} Y_{s_{n-1}} e^{-ipt} e^{ik_x x} = -\frac{i}{k_x \mu_n r_{\beta_n}} e^{-ipt} e^{ik_x x} [A_{21} \left(\frac{\dot{v}}{c}\right)_0 + A_{22} Y_{s_0}]$$

or again with the free surface conditions, and recalling that $\frac{\dot{v}}{c} = \frac{ipv}{c} = ik_x v$

$$(v^d + v^u)_n = -\frac{i}{k_x} e^{-ipt} e^{ik_x x} A_{11} \left(\frac{\dot{v}}{c}\right)_0 = A_{11} v_0 e^{-ipt} e^{ik_x x}$$

$$(v^u - v^d)_n = -\frac{i}{k_x \mu_n r_{\beta_n}} e^{-ipt} e^{ik_x x} A_{21} \left(\frac{\dot{v}}{c}\right)_0 = \frac{1}{\mu_n r_{\beta_n}} A_{21} v_0 e^{-ipt} e^{ik_x x}$$

Now consider the delta excitation in the half space $v_n^u = 1$, then

$$v_n^d + 1 = A_{11} v_0 e^{-ipt} e^{ik_x x}$$

$$1 - v_n^d = \frac{1}{\mu_n r_{\beta_n}} A_{21} v_0 e^{-ipt} e^{ik_x x}$$

Equating and solving for v_n^d

$$\begin{aligned}
\frac{v_n^d + 1}{A_{11}} i &= \mu_n r_{\beta_n} \frac{(1 - v_n^d)}{A_{21}} \\
\frac{v_n^d}{A_{11}} + \frac{1}{A_{11}} &= \frac{\mu_n r_{\beta_n}}{A_{21}} - \mu_n r_{\beta_n} \frac{v_n^d}{A_{21}} \\
\frac{v_n^d}{A_{11}} + \mu_n r_{\beta_n} \frac{v_n^d}{A_{21}} &= v_n^d \left(\frac{1}{A_{11}} + \frac{\mu_n r_{\beta_n}}{A_{21}} \right) = \frac{\mu_n r_{\beta_n}}{A_{21}} - \frac{1}{A_{11}} \\
v_n^d \left(\frac{A_{21} + \mu_n r_{\beta_n} A_{11}}{A_{11} A_{21}} \right) &= \frac{A_{11} \mu_n r_{\beta_n} - A_{21}}{A_{11} A_{21}} \\
v_n^d &= \frac{A_{11} \mu_n r_{\beta_n} - A_{21}}{A_{21} + \mu_n r_{\beta_n} A_{11}}
\end{aligned}$$

Then using $v_n^d + 1 = A_{11} v_0 e^{-ipt} e^{ik_x x}$

$$\frac{A_{11} \mu_n r_{\beta_n} - A_{21}}{A_{21} + \mu_n r_{\beta_n} A_{11}} + 1 = \frac{2 A_{11} \mu_n r_{\beta_n}}{A_{11} \mu_n r_{\beta_n} + A_{21}} = A_{11} v_0 e^{-ipt} e^{ik_x x}$$

Or

$$v_0 = \frac{2 \mu_n r_{\beta_n}}{A_{11} \mu_n r_{\beta_n} + A_{21}} e^{ipt} e^{-ik_x x}$$

To implement we first construct A

$$a_m = \begin{bmatrix} \cos Q_m & \frac{i}{\mu_m r_{\beta_m}} \sin Q_m \\ i \mu_m r_{\beta_m} & \cos Q_m \end{bmatrix} = \begin{bmatrix} a_{11_m} & i a_{12_m} \\ i a_{21_m} & a_{22_m} \end{bmatrix}$$

$$\begin{aligned}
a_{n-1} a_{n-2} &= \begin{bmatrix} a_{11_{n-1}} & i a_{12_{n-1}} \\ i a_{21_{n-1}} & a_{22_{n-1}} \end{bmatrix} \begin{bmatrix} a_{11_{n-2}} & i a_{12_{n-2}} \\ i a_{21_{n-2}} & a_{22_{n-2}} \end{bmatrix} \\
&= \begin{bmatrix} a_{11_{n-1}} a_{11_{n-2}} - a_{12_{n-1}} a_{21_{n-2}} & i(a_{11_{n-1}} a_{12_{n-2}} + a_{12_{n-1}} a_{22_{n-2}}) \\ i(a_{21_{n-1}} a_{11_{n-2}} + a_{22_{n-1}} a_{21_{n-2}}) & a_{22_{n-1}} a_{22_{n-2}} - a_{21_{n-1}} a_{12_{n-2}} \end{bmatrix}
\end{aligned}$$

and so on following this pattern. Then, solve for v_0 as above

$$v_0 = \frac{2\mu_n r_{\beta_n}}{A_{11}\mu_n r_{\beta_n} + A_{21}} e^{ipt} e^{-ik_x x}$$

To recover the displacement recall at any $(m-1)$ interface that

$$\begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_{m-1} \\ Y_{s_{m-1}} \end{bmatrix} = a_{m-1}a_{m-2}\cdots a_0 \begin{bmatrix} \left(\frac{\dot{v}}{c}\right)_0 \\ 0 \end{bmatrix}$$

or since $\frac{\dot{v}}{c} = \frac{ipv}{c} = ik_x v$

$$\begin{bmatrix} v_{m-1} \\ Y_{m-1} \end{bmatrix} = a_{m-1}a_{m-2}\cdots a_0 \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$$

Then the displacement at any point in the layer between interfaces m and $m-1$ is found from

$$\begin{bmatrix} v_m(h) \\ Y_m(h) \end{bmatrix} = a_{m-1}a_m(h) \begin{bmatrix} v_{m-1} \\ Y_{s_{m-1}} \end{bmatrix} = A' \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$$

$$v_m(h) = A'_{11} v_0$$

where $a_m(h)$ is just the a_m evaluated a distance h below the $m-1$ interface. Note that even when $h = d$ we are at interface m. Recalling (3.55) $a_m(h)$ is the case where $Q_m(h) = k_x r_{\beta_m} h$. If r_β is imaginary then

$$r_\beta = -i\sqrt{1 - \frac{c^2}{\beta^2}} \qquad c < \beta$$

$$Q_m(h) = -ik_x h\sqrt{1 - \frac{c^2}{\beta^2}}$$

$$\cos Q_m = \cos\left(-ik_x\sqrt{1 - \frac{c^2}{\beta^2}}h\right)$$

Then using

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{ii\eta} = e^{-\eta} = \cos(i\eta) + i\sin(i\eta) \qquad e^{-i(i\eta)} = e^{\eta} = \cos(i\eta) - i\sin(i\eta)$$

we find

$$\cos(i\eta) = \cosh(\eta) = \frac{e^{-\eta} + e^{\eta}}{2}$$

$$\sin(i\eta) = i\sinh(\eta) = \frac{e^{-\eta} - e^{\eta}}{2i} = i\frac{(e^{\eta} - e^{-\eta})}{2}$$

In the program, when r_β is imaginary we have $r_\beta = ir_2$ and $Q = k_x r_\beta d = ir_2 k_x d = iz_2$

$$\cos Q = \cosh(z_2) = \frac{e^{z_2} + e^{-z_2}}{2} = C_2$$

$$\sin Q = i\sinh(z_2) = i\frac{(e^{z_2} - e^{-z_2})}{2} = iS_2$$

Then

$$a_m = \begin{bmatrix} \cos Q_m & \frac{i}{\mu_m r_{\beta_m}} \sin Q_m \\ i\mu_m r_{\beta_m} \sin Q_m & \cos Q_m \end{bmatrix} = \begin{bmatrix} C_2 & -\frac{1}{\mu r_{\beta_m}} S_2 \\ -i\mu_m r_2 S_2 & C_2 \end{bmatrix} = \begin{bmatrix} C_2 & -\frac{1}{\mu_m i r_2} S_2 \\ -i\mu_m r_2 S_2 & C_2 \end{bmatrix}$$

Thus

$$a_m = \begin{bmatrix} C_2 & \frac{i}{\mu_m r_2} S_2 \\ -i\mu_m r_2 S_2 & C_2 \end{bmatrix}$$

Note the additional minus sign in a_{21}, which we keep track of with the e_2 array in the program.

When calculating v_0 for r_{β_n} real; $r_{\beta_n} = r_{2n}$

$$v_0 = \frac{2\mu_n r_{\beta_n}}{A_{11}\mu_n r_{\beta_n} + A_{11}} = \frac{2\mu_n r_{\beta_n}}{b_{11}\mu_n r_{\beta_n} + ib_{21}} \frac{b_{11}\mu_n r_{\beta_n} - ib_{21}}{b_{11}\mu_n r_{\beta_n} - ib_{21}} = \frac{2b_{11}(\mu_n r_{\beta_n})^2 - i2\mu_n r_{\beta_n} b_{21}}{(b_{11}\mu_n r_{\beta_n})^2 + b_{21}^2}$$

So

$$v_{0_R} = \frac{2b_{11}(\mu_n r_{\beta_n})^2}{(b_{11}\mu_n r_{\beta_n})^2 + b_{21}^2} \qquad v_{0_I} = -\frac{2\mu_n r_{\beta_n} b_{21}}{(b_{11}\mu_n r_{\beta_n})^2 + b_{21}^2}$$

For r_β imaginary and $r_\beta = ir_2$.

$$v_0 = \frac{2\mu_n r_{\beta_n}}{A_{11}\mu_n r_{\beta_n} + A_{21}} = \frac{2i\mu_n r_{2n}}{b_{11} i\mu_n r_{2n} + ib_{21}} = \frac{2\mu_n r_{2n}}{b_{11}\mu_n r_{2n} + b_{21}} = v_{0_R}$$

This is however the half space so r_β will always be real.

3.2.4 Notes on Signs

According to the convention of H53, x is positive in the direction of propagation ($+R$ or 'away') and z is positive down. One would surmise that a positive L direction for P waves would be in the $(-Z, R)$ direction, so that we reverse the sign on Z. If we do this, then a simple crustal model gives $(-Z, R)$ on the transmitted P, or (up, away) as one would expect for the $+L$. The Ps wave is (down, away) which would be a $+H$.

One would then also suspect that a positive incident S wave in the $+H$ direction would be (down, away) $= (+Z, +R)$ so we do not reverse signs in this case. For a simple crustal model the transmitted S is $(+Z, +R)$ as we would expect. The Sp wave is $(+Z, -R)$ or (down, towards) which would make it $-L$.

Note that if we use a 'lithosphere' model (high V over low V) the sense of the converted phases reverses.

For P waves, the principal direction will be the L component, and the dip angle we get from tomhask is shown in figure 3.2. (angle with respect to horizontal; counterclockwise from positive). Note that even though we reversed signs on W, the sense of Z is still positive down; hence:

$$L = -Z \sin\delta + R \cos\delta$$

$$H = Z \cos\delta + R \sin\delta$$

For the S waves, the principal direction will be in the H component, and the dip angle is the compliment of what is shown above $d' = d - 90°$.

$$L = -Z \cos\delta - R \sin\delta$$

$$H = -Z \sin\delta + R \cos\delta$$

3.2.5 Time and Frequency Considerations

We begin with a source time function file, which typically will be a band limited delta function like we find on the L components from SH analysis. From

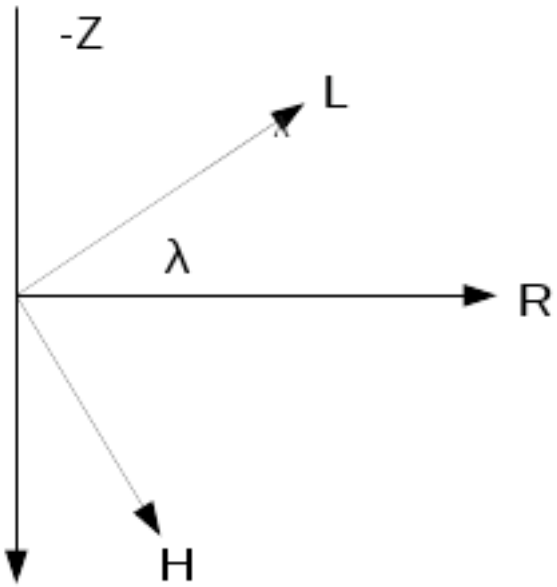

Figure 3.2: Rotation from (Z, R) to (L, H)

this we specify the sampling interval Δ_t and the number of samples n_t. We can also specify a start time, which comes in the header, but for simplicity we will take $t_0 = 0$. We specify where in the source time function file the source time function actually exists with t_{sbeg} and t_{send}. These are times and will be converted to sample numbers i_{sbeg} and i_{isend} in the program. It is this window of the source time function we will use for convolution.

In the current version the actual size of the time series is hardwired into the variable $nn2$ which is set to $MAXPT = 2^{NNPOW}$ where $NNPOW = 16$. Hence the length, in samples, is 65536. For FFT purposes we need a factor of 2 points. The

time duration is then $nn2 \times dt$ (actually $dt \times (nn2 - 1)$), so the frequency interval $\Delta_f = \frac{1}{nn2 \times dt}$. The total number of frequencies for which we need to compute the spectrum should be $MAXPTS = MAXPT/2+1$ (including zero frequency $f = 0$). We need only half because the other half will be the complex conjugate.

We calculate the Thomson Haskell response for each frequency, then convolve with the spectral representations of the source time function by multiplying the spectra. These are then transformed back to the time domain.

3.2.6 Comment on Convolution and Wraparound

In essence each of the two time series has a length $MAXPT$ and so the convolution will have $2 \times MAXPT$ points. Wraparound happens because the FFT assumes periodicity in the signal every $MAXPT$ points. If half of the signal is zero padded then there will be negligible wraparound in the $MAXPT$ that we keep. This is why we we move the source time function to the front of the time series before we compute the spectrum. It also turns out this makes computation of the convolution time delay very simple. We have to guess that all the significant theoretical conversions will arrive within the first half of the time window. This is worth checking from time to time, however, 65536 points seems to be overkill.

3.2.7 Programming Notes

Instead of rewriting the code from scratch, we elected to utilize a similar code for receiver function analysis developed by G.L. Kosarev. However, there is a subtle issue in scaling in the code by Kosarev, and the inferred code from our discussion thus far. We notice the first two rows of his E_n^{-1} matrix (E matrix for the half space) are scaled by $-\alpha^2$ and his last two rows are scaled by $-2\beta^2$ as compared to Haskell. This is a bit odd considering that the rest of the E^{-1} matrices are not similarly scaled. Hence we would have

$$\begin{bmatrix} \Delta_n^d + \Delta_n^u \\ \Delta_n^d - \Delta_n^u \\ \omega_n^d + \omega_n^u \\ \omega_n^d - \omega_n^u \end{bmatrix} = \begin{bmatrix} -\alpha^2 J_{11} & -\alpha^2 J_{12} \\ -\alpha^2 J_{21} & -\alpha^2 J_{22} \\ -2\beta^2 J_{31} & -2\beta^2 J_{32} \\ -2\beta^2 J_{41} & -2\beta^2 J_{42} \end{bmatrix} \begin{bmatrix} \frac{\dot{u}}{c} \\ \frac{\dot{w}}{c} \end{bmatrix}$$

Or better, consider what happens to our evaluation of U and W

$$D = -(iC_{11}C_{32}+C_{21}C_{32})+(C_{11}C_{42}-iC_{21}C_{42})+(iC_{12}C_{31}-C_{22}C_{31})+(C_{12}C_{41}+iC_{22}C_{41})$$

D always involves a row $(1,2)$ term multiplying a $(3,4)$ term, so

$$U_{\text{Kosarev}} = 2\alpha^2\beta^2 D_{\text{Haskell}}$$

For the incident P wave

$$U_{\text{Kosarev}} = -\frac{2\beta^2}{2\beta^2\alpha^2}\frac{2c}{\alpha_n}\frac{(J_{32}-J_{42})}{D} = -\frac{1}{\alpha^2}U_{\text{Haskell}}$$

And for the incident S wave

$$U_{\text{Kosarev}} = -\frac{\alpha^2}{2\alpha^2\beta^2}\frac{c}{\beta_n}\frac{(J_{22}-J_{12})}{\gamma_n r_{\beta_n} D} = -\frac{1}{2\beta^2}U_{\text{Haskell}}$$

and the same logic for W. Again, the ratio $\frac{U}{W}$ is unchanged.

When we do the inversion, we really only care about this ratio. For the P wave problem, we use the Z component as the basis for W, and multiply it by the $\frac{U}{W}$ ratio to get the R component.

When using the code we specify a $1D$ model file, and the original 'L' component which serves as a source time function given in $ASCII$ Q format. We will then specify a window about the original L to isolate the source time function using t_{sbeg}, t_{send}, and t_0. Then we copy the source time window to array z_1 and Fourier transform. Now, Z_1 and Z_2 become the real and imaginary parts of the spectrum.

The emergence angle is computed by

$$L = Z\sin a - R\cos a$$

$$H = -Z\cos a - R\sin a$$

where a is the 'dip' angle with respect to the horizontal plane. Hence, as dip $\rightarrow 90°$, $L \rightarrow Z$, $H \rightarrow -R$ and as dip $\rightarrow 0°$, $L \rightarrow -R$, $H \rightarrow -Z$. Note that this R is positive

towards the source (negative of usual sense). The emergence angle is with respect to the vertical axis. We take 5° in this example, the deduced dip was 83.3°.

3.2.8 Note on Sign of Transforms

When using any other method with the wave tomography software from Pratt we must be mindful of the signs in the Fourier Transform, being Pratt's forward and inverse transform are opposite of convention. In H53 we solve for $F(\omega)$ so that

$$f(t) = \int F(\omega) e^{i(\omega t - kx)} d\omega$$

In the wave tomography software we have the source time function (srcsubs.f)

$$S(\omega) = \int s(t) e^{i\omega t} dt$$

and in eltime, etc to go from frequency to time

$$f(t) = \int F(\omega) e^{-i\omega t} d\omega$$

in the 2.5D derivation we use

$$U = A(\omega) e^{i(\omega t - \omega p_y y - k_x x - k_z z)}$$

which is like H53. But it seems that to be consistent with the wave tomography we should use

$$U^* = A^*(\omega) e^{-i(\omega t - \omega p_y y - k_x x - k_z z)} = A$$

Thus, the frequency domain representations in the wave tomography should be the complex conjugate of the Haskell form. Note, that doesn't matter for $f(t)$ because it is real.

$$f(t) = \int F(\omega) e^{i(\omega t - kx)} d\omega = \int F^*(\omega) e^{-i(\omega t - kx)} d\omega$$

Hence, in explanewv.f we take the conjugate of the H53 and H62 solution. What appears to work is to solve for u and $-w$, take the conjugate in explanewv.f to back

propagate

$$f(t,x,z) = e^{-ik_x x}\int F(\omega,z)e^{i\omega t}d\omega = f^*(t,x,z)$$
$$f^*(t,x,z) = e^{ik_x x}\int F^*(\omega,z)e^{-i\omega t}d\omega$$
$$= e^{i\omega p_x x}\int F^*(\omega)e^{ik_z z}e^{-i\omega t}d\omega$$

where

$$\frac{\omega}{k_x} = c = \frac{1}{p_x}$$

Conjugating reverses the direction of time and space. We can reverse time back again by using $e^{-i\omega t}$ and x with $e^{ik_x x}$, but we reverse z again by transforming F with a negative z. This makes z positive down which is the convention of the wave tomography software.

One outstanding question thus far is why don't we do this for S?

3.2.9 The Fluid Layer

If we make observations near an ocean then we may be interested the effect of a fluid top layer. In this case we can not directly take the limit $\beta_m = 0$ since the inverse of E_m given by (3.18) becomes singular. However, we may define an *effective* inverse transform by setting $\beta_m = 0$ in (3.22) and setting $\tau_{m-1} = 0$ we find

$$F_m^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{\rho_m \alpha_m^2} & 0 \\ 0 & -\frac{c^2}{\alpha_m^2 r_{\alpha_m}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.56}$$

where we notice we only have two linearly independent columns which span our range, which are the unknown coefficients. If we set $\beta_m = 0$ in (3.20) we obtain

$$D_m = \begin{bmatrix} -\frac{\alpha_m^2}{c^2}\cos P_m & i\frac{\alpha_m^2}{c^2}\sin P_m & 0 & 0 \\ i\frac{\alpha_m^2}{c^2}r_{\alpha_m}\sin P_m & -\frac{\alpha_m^2}{c^2}r_{\alpha_m}\cos P_m & 0 & 0 \\ \rho_m\alpha_m^2\cos P_m & -i\rho_m\alpha_m^2\sin P_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.57}$$

from which we multiply

$$a_m = D_m F_m^{-1} = \begin{bmatrix} -\frac{\alpha_m^2}{c^2}\cos P_m & i\frac{\alpha_m^2}{c^2}\sin P_m & 0 & 0 \\ i\frac{\alpha_m^2}{c^2}r_{\alpha_m}\sin P_m & -\frac{\alpha_m^2}{c^2}r_{\alpha_m}\cos P_m & 0 & 0 \\ \rho_m\alpha_m^2\cos P_m & -i\rho_m\alpha_m^2\sin P_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\rho_m\alpha_m^2} & 0 \\ 0 & -\frac{c^2}{\alpha_m^2 r_{\alpha_m}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to obtain

$$a_m = D_m F_m^{-1} = \begin{bmatrix} 0 & -i\frac{1}{r_{\alpha_m}}\sin P_m & -\frac{1}{\rho_m c^2}\cos P_m & 0 \\ 0 & \cos P_m & i\frac{r_{\alpha_m}}{\rho_m c^2}\sin P_m & 0 \\ 0 & i\frac{\rho_m c^2}{r_{\alpha_m}}\sin P_m & \cos P_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.58}$$

We note the difference in the minus sign for entry a_{12}. If the top layer is fluid then

$$E_m \begin{bmatrix} \Delta_m^d + \Delta_m^u \\ \Delta_m^d - \Delta_m^u \\ \omega_m^d - \omega_m^u \\ \omega_m^d + \omega_m^u \end{bmatrix} = \begin{bmatrix} \frac{\dot{u}_{m-1}}{c} \\ \frac{\dot{w}_{m-1}}{c} \\ \sigma_{m-1} \\ \tau_{m-1} \end{bmatrix} = a_{m-1}a_{m-2}\cdots a_0 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = M \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \\ M_{41} & M_{42} \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \end{bmatrix}$$

Applying (3.58) we obtain

$$a_0 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -i\frac{1}{r_{\alpha_0}}\sin P_0 & -\frac{1}{\rho_0 c^2}\cos P_0 & 0 \\ 0 & \cos P_0 & i\frac{r_{\alpha_0}}{\rho_0 c^2}\sin P_0 & 0 \\ 0 & i\frac{\rho_0 c^2}{r_{\alpha_0}}\sin P_0 & \cos P_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i\frac{1}{r_{\alpha_0}}\sin P_0 \frac{\dot{w}_0}{c} \\ \cos P_0 \frac{\dot{w}_0}{c} \\ i\frac{\rho_0 c}{r_{\alpha_0}}\sin P_0 \dot{w}_0 \\ 0 \end{bmatrix}$$

The obvious problem here is there is no solution for u_0, however as noted by H53 an ideal fluid does require continuity of tangential displacement at the boundary, so u_0 is irrelevant. Alternatively, if the stress is zero everywhere on the surface, then there is is no horizontal gradient, i.e. $u_0 = 0$. We must note though that although u is not required to be continuous across the interface, the continuity of vertical stress implicitly creates a u displacement by the pressure gradient in x. If we consider the relationship between stress and pressure direct simplification of (3.11) and (3.15) yield

$$\begin{aligned}\frac{\dot{w}}{c} &= -\frac{\alpha_m^2}{c^2} r_{\alpha_m}[-i(\Delta_m^d + \Delta_m^u)\sin kr_{\alpha_m} z + (\Delta_m^d - \Delta_m^u)\cos kr_{\alpha_m} z] \\ \sigma &= \rho_m \alpha_m^2 [(\Delta_m^d + \Delta_m^u)\cos kr_{\alpha_m} z - i(\Delta_m^d - \Delta_m^u)\sin kr_{\alpha_m} z]\end{aligned}$$

$$\begin{aligned}\frac{\partial \sigma}{\partial z} &= \rho_m \alpha_m^2 kr_{\alpha_m}[-(\Delta_m^d + \Delta_m^u)\sin kr_{\alpha_m} z - i(\Delta_m^d - \Delta_m^u)\cos kr_{\alpha_m} z] \\ &= -\rho_m \alpha_m^2 kr_{\alpha_m}[(\Delta_m^d + \Delta_m^u)\sin kr_{\alpha_m} z + i(\Delta_m^d - \Delta_m^u)\cos kr_{\alpha_m} z]\end{aligned}$$

$$\begin{aligned}\frac{1}{c}\frac{\partial \dot{w}}{\partial t} &= -\frac{\alpha_m^2}{c^2} ipr_{\alpha_m}[-i(\Delta_m^d + \Delta_m^u)\sin kr_{\alpha_m} z + (\Delta_m^d - \Delta_m^u)\cos kr_{\alpha_m} z] \\ &= -\frac{\alpha_m^2}{c^2} pr_{\alpha_m}[(\Delta_m^d + \Delta_m^u)\sin kr_{\alpha_m} z + i(\Delta_m^d - \Delta_m^u)\cos kr_{\alpha_m} z]\end{aligned}$$

Then

$$\rho_m \ddot{w} = -\frac{\rho_m \alpha_m^2}{c} pr_{\alpha_m}[(\Delta_m^d + \Delta_m^u)\sin kr_{\alpha_m} z + i(\Delta_m^d - \Delta_m^u)\cos kr_{\alpha_m} z] = \frac{\partial \sigma}{\partial z}\frac{p}{kc} = \frac{\partial \sigma}{\partial z}$$

Thus,

$$\rho \ddot{w} = -\frac{\partial P}{\partial z}; \qquad P = -\sigma$$

Alternatively, we can proceed from the relationship between pressure and displacement

$$\sigma = \rho\alpha^2\Delta = K\Delta = K\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = K\nabla \cdot \mathbf{u} = -P$$

Similarly, then

$$\rho \ddot{u} = -\frac{\partial P}{\partial x} = \frac{\partial \sigma}{\partial x} = -ik_x P = -ik_x \rho_m \alpha_m^2 [(\Delta_m^d + \Delta_m^u)\cos kr_{\alpha_m} z - i(\Delta_m^d - \Delta_m^u)\sin kr_{\alpha_m} z]$$

so that

$$\begin{aligned}\frac{\dot{u}}{c} &= -i\frac{1}{ip\rho_m c}k_x\rho_m\alpha_m^2[(\Delta_m^d+\Delta_m^u)\cos kr_{\alpha_m z}-i(\Delta_m^d-\Delta_m^u)\sin kr_{\alpha_m z}]\\ &= -\frac{\alpha^2}{c^2}[(\Delta_m^d+\Delta_m^u)\cos kr_{\alpha_m z}-i(\Delta_m^d-\Delta_m^u)\sin kr_{\alpha_m z}]\end{aligned}$$

Which is the expression for u in (3.11). Note that pressure and displacement are not independent, so we choose one, which here will be u instead of stress. So

$$E_0^{-1} = \begin{bmatrix} -\frac{c^2}{\alpha_0^2} & 0 & 0 & 0 \\ 0 & -\frac{c^2}{\alpha_0^2 r_{\alpha_0}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a_0 = D_0E_0^{-1}\begin{bmatrix} -\frac{\alpha_0^2}{c^2}\cos P_0 & i\frac{\alpha_0^2}{c^2}\sin P_0 & 0 & 0 \\ i\frac{\alpha_0^2}{c^2}r_{\alpha_0}\sin P_0 & -\frac{\alpha_0^2}{c^2}r_{\alpha_0}\cos P_0 & 0 & 0 \\ \rho_0\alpha_0^2\cos P_0 & -i\rho_0\alpha_0^2\sin P_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{c^2}{\alpha_0^2} & 0 & 0 & 0 \\ 0 & -\frac{c^2}{\alpha_0^2 r_{\alpha_0}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos P_0 & -i\frac{1}{r_{\alpha_0}}\sin P_0 & 0 & 0 \\ -i\frac{1}{r_{\alpha_0}}\sin P_0 & \cos P_0 & 0 & 0 \\ -\rho_0 c^2\cos P_0 & i\frac{\rho_0 c^2}{r_{\alpha_0}}\sin P_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Returning to the J notation we have

$$J'a_0\begin{bmatrix}\frac{\dot{u}}{c}\\ \frac{\dot{w}}{c}\\ 0\end{bmatrix} = J'\begin{bmatrix} \cos P_0 & -i\frac{1}{r_{\alpha_0}}\sin P_0 & 0 & 0 \\ -i\frac{1}{r_{\alpha_0}}\sin P_0 & \cos P_0 & 0 & 0 \\ -\rho_0 c^2\cos P_0 & i\frac{\rho_0 c^2}{r_{\alpha_0}}\sin P_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}\frac{\dot{u}}{c}\\ \frac{\dot{w}}{c}\\ 0\end{bmatrix}$$

so as before

$$\begin{bmatrix} \Delta_n^d + \Delta_n^u \\ \Delta_n^d - \Delta_n^u \\ \omega_n^d - \omega_n^u \\ \omega_n^d + \omega_n^u \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \\ J_{41} & J_{42} \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \end{bmatrix}$$

as compared to the old form

$$a_0 \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -i\frac{1}{r_{\alpha_0}} \sin P_0 & -\frac{1}{\rho_0 c^2} \cos P_0 & 0 \\ 0 & \cos P_0 & i\frac{r_{\alpha_0}}{\rho_0 c^2} \sin P_0 & 0 \\ 0 & i\frac{\rho_0 c^2}{r_{\alpha_0}} \sin P_0 & \cos P_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_0}{c} \\ \frac{\dot{w}_0}{c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -ir_{\alpha_0} \sin P_0 \frac{\dot{w}_0}{c} \\ \cos P_0 \frac{\dot{w}_0}{c} \\ i\frac{\rho_0 c}{r_{\alpha_0}} \sin P_0 \dot{w}_0 \\ 0 \end{bmatrix}$$

An alternative approach is to solve for stress (pressure) at any depth and then convert this to u and w via the gradient.

CHAPTER 4
Dispersion Analysis

4.1 The Error Functional

For dispersion analysis we wish to determine how much our finite difference forms will cause our numerical group, $\frac{d\omega}{dk}$, and phase, $\frac{\omega}{k}$, velocities to differ from a their analytic counterparts. To develop an analytic representation pertinent to our numerical methods we will assume an isotropic, homogeneous, unbounded, source-less medium.

To proceed we will simplify our non-rotated and rotated continuous equations of motion based on our aforementioned assumptions and express the system as a linear combination of rotated and non rotated forms. We then apply an appropriate discretization to the stiffness terms and mass lump the inertial terms. Expressing this finite system in matrix form we obtain an associated 'eigenvalue' problem in frequency, ω. After substitution the resulting determinant is cubic in frequency, from which we are then able to obtain our desired group and phase velocities. With these expressions for group and phase velocitiy Jo et al, (1996) and seek coefficients a and b as to minimize the numerical dispersion. As an added bonus, this method implicitly reduces numerical anistropy when applied to a $2.5D$ system.

To begin we define a set of functionals to represent error in the L_2 norm for phase velocity (ph)

$$E^P_{ph} = \left[1 - \frac{v^P_{ph}}{\alpha}\right]^2; \qquad E^{SV}_{ph} = \left[1 - \frac{v^{SV}_{ph}}{R\alpha}\right]^2; \qquad E^{SH}_{ph} = \left[1 - \frac{v^{SH}_{ph}}{R\alpha}\right]^2 \tag{4.1}$$

and group velocity (gr)

$$E^P_{gr} = \left[1 - \frac{v^P_{gr}}{\alpha}\right]^2; \qquad E^{SV}_{gr} = \left[1 - \frac{v^{SV}_{gr}}{R\alpha}\right]^2; \qquad E^{SH}_{gr} = \left[1 - \frac{v^{SH}_{gr}}{R\alpha}\right]^2 \tag{4.2}$$

for our compressional P, shear-vertical SV, and shear-horizontal SH waves. Note

we adopt the big R notation of Stekl and Pratt which is related to Poisson's ratio, σ by

$$R = \sqrt{\frac{\frac{1}{2} - \sigma}{1 - \sigma}} = \frac{\beta}{\alpha} \tag{4.3}$$

We formally seek to minimize largest errors over $k^* = \frac{k\Delta}{2\pi} = \frac{1}{G}$ provided $G = \frac{1}{4}$ is the grid points per wavelength, and propagation angle, $\theta \in \left[0, \frac{\pi}{4}\right]$. To make the calculations more intuitive, we change variables and minimize over wavenumber $k \in \left[0, \frac{2\pi}{G}\right]$, hence

$$\frac{1}{2\pi} \min \left[\int_{\epsilon_1}^{2\pi/G} \int_{\epsilon_2}^{\pi/4} \max\{E_{ph}^{P},\, E_{ph}^{SV},\, E_{ph}^{SH},\, E_{gr}^{P},\, E_{gr}^{SV},\, E_{gr}^{SH}\}\, d\theta\, dk \right] \tag{4.4}$$

Because we generate a decoupling in the case when our apparent slowness in y is 0 we choose lower bounds of ϵ_1 and ϵ_2. Numerical tests indicate our our solutions are 'almost' complex as a result of finite precision. As an artifact of our numerical integration we choose $\epsilon_1 = \epsilon_2 = .031$ which proves stable for all cases.

4.1.1 Non Rotated Form

Our simplified equation of motion in three dimensions is given by

$$-\omega^2 \rho u_i = (\lambda + \mu) \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} + \mu \frac{\partial^2 u_i}{\partial^2 x_j} \tag{4.5}$$

or in longer form

$$-\omega^2 \rho u = (\lambda + \mu) \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right] + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

$$-\omega^2 \rho v = (\lambda + \mu) \left[\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z} \right] + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right]$$

$$-\omega^2 \rho w = (\lambda + \mu) \left[\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial z^2} \right] + \mu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right]$$

Then rewriting with our $2.5D$ assumption which yields $\frac{\partial u}{\partial y} = -i\omega p_y u$ and $\frac{\partial u}{\partial y^2} = -(\omega^2 p_y)^2$ we obtain

$$-\omega^2\rho u = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\left(-\omega^2 p_y^2 u + \frac{\partial^2 u}{\partial z^2}\right) + (\lambda+\mu)\frac{\partial^2 w}{\partial x \partial z} - i\omega p_y(\lambda+\mu)\frac{\partial v}{\partial x}$$

$$-\omega^2\rho v = (\lambda+2\mu)(-\omega^2 p_y^2)v + \mu\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial^2 v}{\partial z^2} - i\omega p_y(\lambda+\mu)\frac{\partial u}{\partial x} - i\omega p_y(\lambda+\mu)\frac{\partial w}{\partial z}$$

$$-\omega^2\rho w = (\lambda+2\mu)\frac{\partial^2 w}{\partial z^2} + \mu\left(-\omega^2 p_y^2 w + \frac{\partial^2 w}{\partial x^2}\right) + (\lambda+\mu)\frac{\partial^2 u}{\partial x \partial z} - i\omega p_y(\lambda+\mu)\frac{\partial v}{\partial z}$$

or

$$\omega^2(\mu p_y^2 - \rho)u = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial z^2} + (\lambda+\mu)\frac{\partial^2 w}{\partial x \partial z} - i\omega p_y(\lambda+\mu)\frac{\partial v}{\partial x}$$

$$\omega^2((\lambda+2\mu)p_y^2 - \rho)v = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) - i\omega p_y(\lambda+\mu)\frac{\partial u}{\partial x} - i\omega p_y(\lambda+\mu)\frac{\partial w}{\partial z}$$

$$\omega^2(\mu p_y^2 - \rho)w = (\lambda+2\mu)\frac{\partial^2 w}{\partial z^2} + \mu\frac{\partial^2 w}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u}{\partial x \partial z} - i\omega p_y(\lambda+\mu)\frac{\partial v}{\partial z}$$

Then dividing by ρ and recalling that $\alpha^2 = \frac{\lambda+2\mu}{\rho}$ and $\beta^2 = \frac{\mu}{\rho}$ we have

$$\omega^2(\beta^2 p_y^2 - 1)u = \alpha^2\frac{\partial^2 u}{\partial x^2} + \beta^2\frac{\partial^2 u}{\partial z^2} + (\alpha^2-\beta^2)\frac{\partial^2 w}{\partial x \partial z} - i\omega p_y(\alpha^2-\beta^2)\frac{\partial v}{\partial x}$$

$$\omega^2(\alpha^2 p_y^2 - 1)v = \beta^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) - i\omega p_y(\alpha^2-\beta^2)\frac{\partial u}{\partial x} - i\omega p_y(\alpha^2-\beta^2)\frac{\partial w}{\partial z}$$

$$\omega^2(\beta^2 p_y^2 - 1)w = \alpha^2\frac{\partial^2 w}{\partial z^2} + \beta^2\frac{\partial^2 w}{\partial x^2} + (\alpha^2-\beta^2)\frac{\partial^2 u}{\partial x \partial z} - i\omega p_y(\alpha^2-\beta^2)\frac{\partial v}{\partial z}$$

Finally, introducing the big R notation from (4.3), where $R^2 = \frac{\beta^2}{\alpha^2}$, and dividing out by α^2

$$\omega^2(R^2 p_y^2 - \frac{1}{\alpha^2})u = \frac{\partial^2 u}{\partial x^2} + R^2\frac{\partial^2 u}{\partial z^2} + (1-R^2)\frac{\partial^2 w}{\partial x \partial z} - i\omega p_y(1-R^2)\frac{\partial v}{\partial x}$$

$$\omega^2(p_y^2 - \frac{1}{\alpha^2})v = R^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) - i\omega p_y(1-R^2)\frac{\partial u}{\partial x} - i\omega p_y(1-R^2)\frac{\partial w}{\partial z}$$

$$\omega^2(R^2 p_y^2 - \frac{1}{\alpha^2})w = \frac{\partial^2 w}{\partial z^2} + R^2\frac{\partial^2 w}{\partial x^2} + (1-R^2)\frac{\partial^2 u}{\partial x \partial z} - i\omega p_y(1-R^2)\frac{\partial v}{\partial z}$$

Defining

$$T_1 \equiv R^2 p_y^2 - \frac{1}{\alpha^2} \tag{4.11a}$$

$$T_2 \equiv p_y^2 - \frac{1}{\alpha^2} \tag{4.11b}$$

we at last obtain

$$\omega^2 T_1 u = \frac{\partial^2 u}{\partial x^2} + R^2 \frac{\partial^2 u}{\partial z^2} + (1 - R^2) \frac{\partial^2 w}{\partial x \partial z} - i\omega p_y (1 - R^2) \frac{\partial v}{\partial x} \tag{4.12a}$$

$$\omega^2 T_2 v = R^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right) - i\omega p_y (1 - R^2) \frac{\partial u}{\partial x} - i\omega p_y (1 - R^2) \frac{\partial w}{\partial z} \tag{4.12b}$$

$$\omega^2 T_1 w = \frac{\partial^2 w}{\partial z^2} + R^2 \frac{\partial^2 w}{\partial x^2} + (1 - R^2) \frac{\partial^2 u}{\partial x \partial z} - i\omega p_y (1 - R^2) \frac{\partial v}{\partial z} \tag{4.12c}$$

4.1.2 Rotated Equations

Again, if we rotated our system 45°, we can relate our original (x, y) frame to (x', y') via

$$x' = \tfrac{x+z}{\sqrt{2}}; \quad z' = \tfrac{z-x}{\sqrt{2}}$$

and back by

$$x = \tfrac{x'-z'}{\sqrt{2}}; \quad z = \tfrac{x'+z'}{\sqrt{2}}$$

Taking the first derivatives we recover (1.14a) and (1.14b) given by

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial x} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial z'} \right) \tag{4.13a}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x'} \frac{\partial x'}{\partial z} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial z} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial z'} \right) \tag{4.13b}$$

The second order derivatives given by

$$\frac{\partial^2}{\partial x^2} = \frac{1}{2} \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial z'^2} \right) - \frac{\partial^2}{\partial x' \partial z'} \tag{4.14a}$$

$$\frac{\partial^2}{\partial z^2} = \frac{1}{2} \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial z'^2} \right) + \frac{\partial^2}{\partial x' \partial z'} \tag{4.14b}$$

$$\frac{\partial^2}{\partial x \partial z} = \frac{1}{2}\left(\frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial z'^2}\right) \tag{4.14c}$$

We can then rewrite the equations of motion in the original frame as

$$\omega^2(\beta^2 p_y^2 - 1)u = \frac{\alpha^2}{2}\left(\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial z'^2}\right) + \frac{\beta^2}{2}\left(\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial z'^2}\right) - (\alpha^2 - \beta^2)\frac{\partial^2 u}{\partial x' \partial z'} + \frac{(\alpha^2 - \beta^2)}{2}\left(\frac{\partial^2 w}{\partial x'^2} - \frac{\partial^2 w}{\partial z'^2}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial v}{\partial x'} - \frac{\partial v}{\partial z'}\right)$$

$$\omega^2(\alpha^2 p_y^2 - 1)v = \beta^2\left(\frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial z'^2}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial w}{\partial x'} + \frac{\partial w}{\partial z'}\right)$$

$$\omega^2(\beta^2 p_y^2 - 1)w = \frac{\alpha^2}{2}\left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial z'^2}\right) + \frac{\beta^2}{2}\left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial z'^2}\right) + (\alpha^2 - \beta^2)\frac{\partial^2 w}{\partial x' \partial z'} + \frac{(\alpha^2 - \beta^2)}{2}\left(\frac{\partial^2 u}{\partial x'^2} - \frac{\partial^2 u}{\partial z'^2}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial v}{\partial x'} + \frac{\partial v}{\partial z'}\right)$$

or slightly reduced to

$$\omega^2(\beta^2 p_y^2 - 1)u = \frac{(\alpha^2 + \beta^2)}{2}\left(\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial z'^2}\right) - (\alpha^2 - \beta^2)\frac{\partial^2 u}{\partial x' \partial z'} + \frac{(\alpha^2 - \beta^2)}{2}\left(\frac{\partial^2 w}{\partial x'^2} - \frac{\partial^2 w}{\partial z'^2}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial v}{\partial x'} - \frac{\partial v}{\partial z'}\right)$$

$$\omega^2(\alpha^2 p_y^2 - 1)v = \beta^2\left(\frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial z'^2}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial w}{\partial x'} + \frac{\partial w}{\partial z'}\right)$$

$$\omega^2(\beta^2 p_y^2 - 1)w = \frac{(\alpha^2 + \beta^2)}{2}\left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial z'^2}\right) + (\alpha^2 - \beta^2)\frac{\partial^2 w}{\partial x' \partial z'} + \frac{(\alpha^2 - \beta^2)}{2}\left(\frac{\partial^2 u}{\partial x'^2} - \frac{\partial^2 u}{\partial z'^2}\right) - i\omega p_y \frac{(\alpha^2 - \beta^2)}{\sqrt{2}}\left(\frac{\partial v}{\partial x'} + \frac{\partial v}{\partial z'}\right)$$

Then with our expressions for T_1 and T_2 given by (4.11a) and (4.11b), dividing out by α^2 and recalling our big R notation we find our rotated forms are given by

$$\omega^2 T_1 u = \frac{(1+R^2)}{2}\left(\frac{\partial^2 u}{\partial x'^2}+\frac{\partial^2 u}{\partial z'^2}\right) - (1-R^2)\frac{\partial^2 u}{\partial x'\partial z'} + \frac{(1-R^2)}{2}\left(\frac{\partial^2 w}{\partial x'^2}-\frac{\partial^2 w}{\partial z'^2}\right) - i\omega p_y \frac{(1-R^2)}{\sqrt{2}}\left(\frac{\partial v}{\partial x'}-\frac{\partial v}{\partial z'}\right) \tag{4.17a}$$

$$\omega^2 T_2 v = R^2\left(\frac{\partial^2 v}{\partial x'^2}+\frac{\partial^2 v}{\partial z'^2}\right) - i\omega p_y \frac{(1-R^2)}{\sqrt{2}}\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right) - i\omega p_y \frac{(1-R^2)}{\sqrt{2}}\left(\frac{\partial w}{\partial x'}+\frac{\partial w}{\partial z'}\right) \tag{4.17b}$$

$$\omega^2 T_1 w = \frac{(1+R^2)}{2}\left(\frac{\partial^2 w}{\partial x'^2}+\frac{\partial^2 w}{\partial z'^2}\right) + (1-R^2)\frac{\partial^2 w}{\partial x'\partial z'} + \frac{(1-R^2)}{2}\left(\frac{\partial^2 u}{\partial x'^2}-\frac{\partial^2 u}{\partial z'^2}\right) - i\omega p_y \frac{(1-R^2)}{\sqrt{2}}\left(\frac{\partial v}{\partial x'}+\frac{\partial v}{\partial z'}\right) \tag{4.17c}$$

We can then express our our continuous equation of motion as a linear combination of non-rotated and rotated differential forms

$$\omega^2 T_1 u = aA_1 + (1-a)A_2 \tag{4.18a}$$

$$\omega^2 T_2 v = aB_1 + (1-a)B_2 \tag{4.18b}$$

$$\omega^2 T_1 w = aC_1 + (1-a)C_2 \tag{4.18c}$$

where A_1 represents the spatial operators in (4.12a) and A_2 represents the spatial operators in (4.17a) and so forth.

Next we can spatially average to the inertial term, whose significance in the finite element community is to diagonalize the mass matrix. For both finite elements and finite differences it appears to increase convergence. Now our approximation

looks like

$$u \approx bu_{i,k} + \frac{(1-b)}{4}(u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1}) \tag{4.19a}$$

$$v \approx bv_{i,k} + \frac{(1-b)}{4}(v_{i+1,k} + v_{i-1,k} + v_{i,k+1} + v_{i,k-1}) \tag{4.19b}$$

$$w \approx bw_{i,k} + \frac{(1-b)}{4}(w_{i+1,k} + w_{i-1,k} + w_{i,k+1} + w_{i,k-1}) \tag{4.19c}$$

Then if we combine (4.18a), (4.18b), (4.18c) with (4.19a), (4.19b), (4.19c) we arrive at

$$\omega^2 T_1 \left(bu_{i,k} + \frac{(1-b)}{4}(u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1}) \right) = aA_1 + (1-a)A_2 \tag{4.20a}$$

$$\omega^2 T_2 \left(bv_{i,k} + \frac{(1-b)}{4}(v_{i+1,k} + v_{i-1,k} + v_{i,k+1} + v_{i,k-1}) \right) = aB_1 + (1-a)B_2 \tag{4.20b}$$

$$\omega^2 T_1 \left(bw_{i,k} + \frac{(1-b)}{4}(w_{i+1,k} + w_{i-1,k} + w_{i,k+1} + w_{i,k-1}) \right) = aC_1 + (1-a)C_2 \tag{4.20c}$$

4.1.3 Finite Difference Forms and the von Neumann Method

To proceed with (4.20a), (4.20b), and (4.20c) we want to create a ratio between the numerical and theoretic case and thereby must discretize the problem. Our finite difference forms are given by

$$f^+ = f_{i+1,k}; \qquad f^- = f_{i-1,k}; \qquad f^+_+ = f_{i+1,k+1}; \qquad f_+ = f_{i,k+1};$$
$$f_- = f_{i,k-1}; \qquad f^-_- = f_{i-1,k-1}; \qquad f^+_- = f_{i+1,k-1}; \qquad f^-_+ = f_{i-1,k+1}$$

or visually

$$\begin{matrix} f^-_- & f_- & f^+_- \\ f^- & f & f^+ \\ f^-_+ & f_+ & f^+_+ \end{matrix}$$

Then writing out the stencils we have

$$\frac{\partial f}{\partial x} = \frac{f^+ - f^-}{2\Delta} \tag{4.21a}$$
$$\frac{\partial f}{\partial z} = \frac{f_+ - f_-}{2\Delta} \tag{4.21b}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{f^+ - 2f + f^-}{\Delta^2} \tag{4.21c}$$
$$\frac{\partial^2 f}{\partial z^2} = \frac{f_+ - 2f + f_-}{\Delta^2} \tag{4.21d}$$
$$\frac{\partial^2 f}{\partial x \partial z} = \frac{f^+_+ - f^+_- - f^-_+ + f^-_-}{4\Delta^2} \tag{4.21e}$$
$$\frac{\partial f}{\partial x'} = \frac{f^+_+ - f^-_-}{2\sqrt{2}\Delta} \tag{4.21f}$$
$$\frac{\partial f}{\partial z'} = \frac{f^-_+ - f^+_-}{2\sqrt{2}\Delta} \tag{4.21g}$$
$$\frac{\partial^2 f}{\partial x'^2} = \frac{f^+_+ - 2f + f^-_-}{2\Delta^2} \tag{4.21h}$$
$$\frac{\partial^2 f}{\partial z'^2} = \frac{f^-_+ - 2f + f^+_-}{2\Delta^2} \tag{4.21i}$$
$$\frac{\partial^2 f}{\partial x' \partial z'} = \frac{1}{\sqrt{2}\Delta}\left(\frac{f_+ - f_-}{\sqrt{2}\Delta} - \frac{f^+ - f_-}{\sqrt{2}\Delta}\right) = \frac{f_+ - f^+ - f^- + f_-}{2\Delta^2} \tag{4.21j}$$

As typical we proceed with a von Neumann analysis by introducing a plane wave solution $(u_0, v_0, w_0)e^{-i(k_x x+k_z z)}$. These are related to the finite difference stencils by

$$f_-^- = A(\omega)e^{i(\omega t-\omega p_y y-k_x(x-\Delta)-k_z(z-\Delta))} = fe^{i(k_x+k_z)\Delta} \tag{4.22a}$$

$$f_- = A(\omega)e^{i(\omega t-\omega p_y y-k_x x-k_z(z-\Delta))} = fe^{ik_z\Delta} \tag{4.22b}$$

$$f_-^+ = A(\omega)e^{i(\omega t-\omega p_y y-k_x(x+\Delta)-k_z(z-\Delta))} = fe^{-i(k_x-k_z)\Delta} \tag{4.22c}$$

$$f^- = A(\omega)e^{i(\omega t-\omega p_y y-k_x(x-\Delta)-k_z z)} = fe^{ik_x\Delta} \tag{4.22d}$$

$$f = A(\omega)e^{i(\omega t-\omega p_y y-k_x x-k_z z)} \tag{4.22e}$$

$$f^+ = A(\omega)e^{i(\omega t-\omega p_y y-k_x(x+\Delta)-k_z z)} = fe^{-ik_x\Delta} \tag{4.22f}$$

$$f_+^- = A(\omega)e^{i(\omega t-\omega p_y y-k_x(x-\Delta)-k_z(z+\Delta))} = fe^{i(k_x-k_z)\Delta} \tag{4.22g}$$

$$f_+ = A(\omega)e^{i(\omega t-\omega p_y y-k_x x-k_z(z+\Delta))} = fe^{-ik_z\Delta} \tag{4.22h}$$

$$f_+^+ = A(\omega)e^{i(\omega t-\omega p_y y-k_x(x+\Delta)-k_z z(z+\Delta))} = fe^{-i(k_x+k_z)\Delta} \tag{4.22i}$$

The discrete forms of (4.20a), (4.20b), and (4.20c) can be written in matrix form as

$$\frac{\omega^2}{\alpha^2}\begin{bmatrix} T_1M_1 & 0 & 0 \\ 0 & T_2M_1 & 0 \\ 0 & 0 & T_1M_1 \end{bmatrix}\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \frac{K_{11}}{\Delta^2} & \frac{\omega p_y K_{21}}{\Delta} & \frac{K_{31}}{\Delta^2} \\ \frac{\omega p_y K_{21}}{\Delta} & \frac{K_{22}}{\Delta^2} & \frac{\omega p_y K_{32}}{\Delta} \\ \frac{K_{31}}{\Delta^2} & \frac{\omega p_y K_{32}}{\Delta} & \frac{K_{33}}{\Delta^2} \end{bmatrix}\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \tag{4.23}$$

This is closely related to the eigenvalue problem in ω for 3D, however we have added eigenvalues into the stiffness terms. As we would expect though, the eigenvalues are in general all real valued with the exception of a few pathological cases when we are at low wavenumber and low θ or we decouple the system with $p_y = 0$, in which case the effect of round off error is quite severe, and thus resulting in 'almost' complex eigenvalues. We can explicitly define our mass and stiffness terms by

$$\begin{aligned} M_1 &= bu_{i,k} + \frac{1-b}{4}(u^+ + u^- + u_+ + u_-) \\ &= b + \frac{1-b}{4}(e^{ik_x\Delta} + e^{-ik_x\Delta} + e^{ik_z\Delta} + e^{-ik_z\Delta}) \\ &= b + \frac{1-b}{4}(2\cos k_x\Delta + 2\cos k_z\Delta) \end{aligned}$$

which simplifies to

$$M_1 = b + \frac{(1-b)}{2}(\cos k_x\Delta + \cos k_z\Delta)$$

Then we note that $k_x\Delta = k\Delta\cos\theta$ and $k_z = k\Delta\sin\theta$. Hence

$$M_1 = b + \frac{(1-b)}{2}(\cos(k\Delta\cos\theta) + \cos(k\Delta\sin\theta) \tag{4.24}$$

with corresponding derivative

$$\frac{dM_1}{dk} = -\frac{\Delta(1-b)}{2}[\sin(k\Delta\cos\theta)\cos\theta + \sin(k\Delta\sin\theta)\sin\theta] \tag{4.25}$$

$$\begin{aligned}
K_{11} =& a\left[\frac{\partial^2 u}{\partial x^2} + R^2\frac{\partial^2 u}{\partial z^2}\right] + (1-a)\left[\frac{(1+R^2)}{2}\left(\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial z'^2}\right) - (1-R^2)\frac{\partial^2 u}{\partial x'\partial z'}\right] \\
=& a\left[u^+ - 2u + u^- + R^2(u_+ - 2u + u_-)\right] \\
&+ \frac{(1-a)}{2}\left[\frac{(1+R^2)}{2}(u_+^+ + u_-^- - 4u + u_+^- + u_-^+) - (1-R^2)(u_+ + u_- - (u^+ + u^-))\right] \\
=& a[e^{ik_x\Delta} + e^{-ik_x\Delta} - 2 + R^2(e^{ik_x\Delta} + e^{-ik_x\Delta} - 2)] \\
&+ \frac{(1-a)}{2}\left[\frac{(1+R^2)}{2}(e^{-i(k_x+k_z)\Delta} + e^{i(k_x+k_z)\Delta} + e^{i(k_x-k_z\Delta)} + e^{-i(k_x-k_z)\Delta} - 4)\right] \\
&- \frac{(1-a)}{2}\left[-\frac{1-R^2}{2}(e^{-ik_z\Delta} + e^{ik_z\Delta} - (e^{-ik_x\Delta} + e^{ik_x\Delta})\right] \\
=& a\left[2\cos k_x\Delta - 2 + 2R^2(\cos k_z\Delta - 1)\right] \\
&+ \frac{(1-a)}{2}\left[\frac{(1+R^2)}{2}(2\cos(k_x+k_z)\Delta + 2\cos(k_x-k_z)\Delta - 4)\right] \\
&- \frac{(1-a)}{2}\left[-\frac{1-R^2}{2}(2\cos k_z\Delta - 2\cos k_x\Delta)\right]
\end{aligned}$$

which simplifies to

$$\begin{aligned}
K_{11} =& 2a\left[\cos k_x\Delta - 1 + R^2(\cos k_z\Delta - 1)\right] \\
&+ (1-a)\left[(1+R^2)(\cos k_x\Delta\cos k_z\Delta - 1) - (1-R^2)(\cos k_z\Delta - \cos k_x\Delta)\right]
\end{aligned}$$

or

$$K_{11} = 2a\left[\cos(k\Delta\cos\theta) - 1 + R^2(\cos(k\Delta\sin\theta) - 1)\right] + (1-a)[(1+R^2)(\cos(k\Delta\cos\theta)\cos(k\Delta\sin\theta) - 1) - (1-R^2)(\cos(k\Delta\sin\theta) - \cos(k\Delta\cos\theta))] \quad (4.26)$$

with corresponding derivative

$$\frac{dK_{11}}{dk} = -\Delta 2a\left[\sin(k\Delta\cos\theta)\Delta\cos\theta + R^2\sin(k\Delta\sin\theta)\sin\theta\right] -\Delta(1-a)(1+R^2)(\sin(k\Delta\cos\theta)\cos(k\Delta\sin\theta)\cos\theta+\cos(k\Delta\cos\theta)\sin(k\Delta\sin\theta)\sin\theta) + \Delta(1-a)(1-R^2)(\sin(k\Delta\sin\theta)\sin\theta - \sin(k\Delta\cos\theta)\cos\theta) \quad (4.27)$$

$$K_{21} = -i(1-R^2)\left[a\frac{dv}{dx} - \frac{(1-a)}{\sqrt{2}}\left(\frac{dv}{dx'} - \frac{dv}{dz'}\right)\right] = \frac{i}{2}(1-R^2)\left[a(v^+ - v-) + \frac{1-a}{2}(v_+^+ - v_-^- - v_+^- + v_-^+)\right] = \frac{-i}{2}(1-R^2) \left[a(e^{-ik_x\Delta} - e^{ik_x\Delta} + \frac{(1-a)}{2}(e^{-i(k_x+k_z)\Delta} - e^{i(k_x+k_z)\Delta} + e^{-i(k_x-k_z)\Delta} - e^{i(k_x-k_z)\Delta})\right] = -\frac{ip_y}{2\Delta}a\left[-2i\sin k_x\Delta + i(1-a)(\sin(k_x+k_z)\Delta + \sin(k_x-k_z)\Delta\right]$$

which simplifies to

$$K_{21} = -(1-R^2)\left[a\sin k_x\Delta + (1-a)\sin k_x\Delta\cos k_z\Delta\right]$$

or

$$K_{21} = -(1-R^2)\left[a\sin(k\Delta\cos\theta) + (1-a)\sin(k\Delta\cos\theta)\cos(k\Delta\sin\theta)\right] \quad (4.28)$$

with corresponding derivative

$$\frac{dK_{21}}{dk} = -\Delta(1-R^2)[a\cos(k\Delta\cos\theta)\cos\theta + (1-a)(\cos(k\Delta\cos\theta)\cos(k\Delta\sin\theta)\cos\theta - \sin(k\Delta\cos\theta)\sin(k\Delta\sin\theta)\sin\theta)] \tag{4.29}$$

$$\begin{aligned} K_{31} &= a(1-R^2)\frac{\partial^2 w}{\partial x \partial z} + (1-a)\frac{1-R^2}{2}\left(\frac{\partial^2 w}{\partial x'} - \frac{\partial^2 w}{\partial z'^2}\right) \\ &= \frac{(1-R^2)}{4}(a(w_+^+ - w_-^+ - w_+^- + w_-^-) + (1-a)(w_+^+ + w_-^- - w_+^- - w_-^+) \\ &= \frac{(1-R^2)}{4}(w_+^+ - w_-^+ - w_+^- + w_-^-) \\ &= \frac{(1-R^2)}{4}(e^{-i(k_x+k_z)\Delta} + e^{i(k_x+k_z)\Delta} - (e^{-i(k_x-k_z)\Delta} + e^{i(k_x-k_z)\Delta})) \\ &= \frac{(1-R^2)}{4}[2\cos(k_x+k_z)\Delta - 2\cos(k_x-k_z)\Delta] \end{aligned}$$

which simplifies to

$$K_{31} = -(1-R^2)\sin(k\Delta\cos\theta)\sin(k\Delta\sin\theta) \tag{4.30}$$

with corresponding derivative

$$\frac{dK_{31}}{dk} = -\Delta(1-R^2)[\cos(k\Delta\cos\theta)\sin(k\Delta\sin\theta)\cos\theta + \sin(k\Delta\cos\theta)\cos(k\Delta\sin\theta)\sin\theta] \tag{4.31}$$

$$\begin{aligned} K_{22} =& R^2\left[a\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) + (1-a)\left(\frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial z'^2}\right)\right] \\ =& R^2\left[a(v^+ + v^- + v_+ + v_- - 4v) + \frac{(1-a)}{2}(v_+^+ + v_-^+ + v_+^- + v_-^- - 4v)\right] \\ =& R^2 a(e^{-ik_x\Delta} + e^{ik_x\Delta} + e^{-ik_z\Delta} + e^{ik_z\Delta} - 4) \\ &+ \frac{R^2}{2}(1-a)(e^{-i(k_x+k_z)\Delta} + e^{i(k_x+k_z)\Delta} + e^{-i(k_x-k_z)\Delta} + e^{-i(k_x-k_z)\Delta} - 4) \\ =& R^2[2a(\cos k_x\Delta + \cos k_z\Delta - 2) + (1-a)(\cos(k_x+k_z)\Delta + \cos(k_x-k_z)\Delta - 2)] \end{aligned}$$

which simplifies to

$$K_{22} = 2R^2\left[a(\cos k_x\Delta + \cos k_z\Delta - 2) + (1-a)(\cos k_x\Delta \cos k_z\Delta - 1)\right]$$

or

$$K_{22} = 2R^2[a(\cos(k\Delta\cos\theta)+\cos(k\Delta\sin\theta)-2)+(1-a)(\cos(k\Delta\cos\theta)\cos(k\Delta\sin\theta)-1)] \tag{4.32}$$

with derivative

$$\begin{aligned}\frac{dK_{22}}{dk} &= -\Delta 2R^2[a(\sin(k\Delta\cos\theta)\cos\theta + \sin(k\Delta\sin\theta)\sin\theta) \\ &\quad + (1-a)(\sin(k\Delta\cos\theta)\cos(k\Delta\sin\theta)\cos\theta + \cos(k\Delta\cos\theta)\sin(k\Delta\sin\theta)\sin\theta)]\end{aligned} \tag{4.33}$$

$$\begin{aligned}K_{32} &= -i(1-R^2)\left(a\frac{\partial w}{\partial z} + \frac{(1-a)}{\sqrt{2}}\left(\frac{\partial w}{\partial x'} + \frac{\partial w}{\partial z'}\right)\right) \\ &= -\frac{i(1-R^2)}{2}\left(a(w_+ - w_-) + \frac{1-a}{2}(w_+^+ - w_-^- + w_+^- - w_-^+\right) \\ &= -\frac{i(1-R^2)}{2} \\ &\left(a(e^{ik_z\Delta} - e^{-ik_z\Delta}) + \frac{(1-a)}{2}(e^{-i(k_x+k_z)\Delta} - e^{i(k_x+k_z)\Delta} + e^{-i(k_x-k_z)\Delta} - e^{i(k_x-k_z)\Delta})\right) \\ &= -i(1-R^2)[-i2a\sin k_z\Delta + i\frac{(1-a)}{2}(-2\sin(k_x+k_x)\Delta + 2\sin(k_x-k_z)\Delta)]\end{aligned}$$

which simplifies to

$$K_{32} = -(1-R^2)(a\sin k_z\Delta - (1-a)\sin k_z\Delta\cos k_x\Delta)$$

or

$$K_{32} = -(1-R^2)(a\sin(k\Delta\sin\theta) - (1-a)\sin(k\Delta\sin\theta)\cos(k\Delta\cos\theta)) \tag{4.34}$$

with derivative

$$\frac{dK_{32}}{dk} = -\Delta(1-R^2)[a\cos(k\Delta\sin\theta)\sin\theta - (1-a)[\cos(k\Delta\sin\theta)\cos(k\Delta\cos\theta)\sin\theta - \sin(k\Delta\sin\theta)\sin(k\Delta\cos\theta)\cos\theta)] \quad (4.35)$$

$$\begin{aligned}
K_{33} =& a\left(\frac{\partial^2 w}{\partial z^2} + R^2\frac{\partial^2 w}{\partial x^2}\right) + (1-a)\left(\frac{(1+R^2)}{2}\left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial z'^2}\right) + (1-R^2)\frac{\partial^2 w}{\partial x'\partial z'}\right) \\
=& a\left[w_+ - 2w + w_- + R^2(w^+ - 2w + w^-)\right] \\
&+ \frac{(1-a)}{2}\left[(1+R^2)(w_+^+ + w_-^- + w_+^- + w_-^+ - 4w) + (1-R^2)(w_+ - w^+ - w^- + w_-)\right] \\
=& a\left[e^{-ik_z\Delta} + e^{ik_z\Delta} - 2 + R^2(e^{-ik_x\Delta} + e^{ik_x\Delta} - 2)\right] \\
&+ \frac{(1-a)}{2}\frac{(1+R^2)}{2}\left[(e^{-i(k_x+k_z)\Delta} + e^{i(k_x+k_z)\Delta} + e^{-i(k_x-k_z)\Delta} + e^{i(k_x-k_z)\Delta} - 4)\right] \\
&+ \frac{(1-a)}{2}(1-R^2)\left(e^{-ik_z\Delta} + e^{ik_z\Delta} - (e^{-ik_x\Delta} + e^{ik_x\Delta})\right) \\
=& a\left[2\cos k_z\Delta - 2 + 2R^2(\cos k_x\Delta - 1)\right] \\
&+ \frac{(1-a)}{2}\frac{(1+R^2)}{2}\left[2\cos(k_x+k_z)\Delta + 2\cos(k_x-k_z)\Delta - 4)\right] \\
&+ \frac{(1-a)}{2}(1-R^2)\left(2\cos k_z\Delta - 2\cos k_x\Delta\right)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
K_{33} =& 2a\left[\cos k_z\Delta - 1 + R^2(\cos k_x\Delta - 1)\right] \\
&+ (1-a)\left[(1+R^2)(\cos k_x\Delta\cos k_z\Delta - 1) + (1-R^2)(\cos k_z\Delta - \cos k_x\Delta)\right]
\end{aligned}$$

or

$$\begin{aligned}
K_{33} &= 2a\left[\cos(k\Delta\sin\theta) - 1 + R^2(\cos(k\Delta\cos\theta) - 1)\right] \\
&+(1-a)[(1+R^2)(\cos(k\Delta\cos\theta)(k\Delta\sin\theta) - 1) \\
&+(1-R^2)(\cos(k\Delta\sin\theta) - \cos(k\Delta\cos\theta))]
\end{aligned} \quad (4.36)$$

and derivative

$$\frac{dK_{33}}{dk} = -\Delta 2a[\sin(k\Delta\sin\theta)\sin\theta + R^2\sin(k\Delta\cos\theta)\cos\theta]$$
$$-\Delta(1-a)(1+R^2)[\sin(k\Delta\cos\theta)\cos(k\Delta\sin\theta)\cos\theta+\cos(k\Delta\cos\theta)\sin(k\Delta\sin\theta)\sin\theta]$$
$$-\Delta(1-a)(1-R^2)[\sin(k\Delta\sin\theta)\sin\theta-\sin(k\Delta\cos\theta)\cos\theta] \quad (4.37)$$

Now identifying the eigenvalue problem in ω gives us the form

$$\omega^6 T_1^2 T_2 M_1^3 - \frac{\omega^4}{\Delta^2}\left[T_1T_2M_1^2(K_{11}+K_{33}) + T_1^2M_1^2K_{22} + p_y^2T_1M_1(K_{21}^2+K_{32}^2)\right]$$
$$+\frac{\omega^2}{\Delta^4}\big[T_1M_1(K_{11}K_{22}+K_{22}K_{33})$$
$$+T_2M_1(K_{11}K_{33}-K_{31}^2)+p_y^2(K_{32}^2K_{11}+K_{21}^2K_{33}-2K_{31}K_{21}K_{32})\big]$$
$$+\frac{1}{\Delta^6}\left[K_{31}^2K_{22}-K_{11}K_{22}K_{33}\right]=0$$

or rearranging

$$(\omega\Delta)^6 - (\omega\Delta)^4\left[\frac{(K_{11}+K_{33})}{T_1M_1}+\frac{K_{22}}{T_2M_1}+p_y^2\right]$$
$$+(\omega\Delta)^2\left[\frac{(K_{11}K_{22}+K_{22}K_{33})}{T_1T_2M_1^2}+\frac{(K_{11}K_{33}-K_{31}^2)}{T_1^2M_1^2}+\frac{(K_{32}^2K_{11}+K_{21}^2K_{33}-2K_{31}K_{21}K_{32})}{T_1^2T_2M_1^3}\right]$$
$$+\frac{K_{31}^2K_{22}-K_{11}K_{22}K_{33}}{T_1^2T_2M_1^3}=0 \quad (4.38)$$

hence we are left with a problem of form the

$$(\omega\Delta)^6 + a\,(\omega\Delta)^4 + b\,(\omega\Delta)^2 + c = 0 \quad (4.39)$$

where I have defined

$$a \equiv -\frac{K_{11}+K_{33}}{T_1M_1} - \frac{K_{22}}{T_2M_1} - p_y^2\frac{(K_{21}^2+K_{32}^2)}{T_1T_2M_1^2} \quad (4.40a)$$

$$\frac{da}{dk} = -\frac{(K'_{11}+K'_{33})}{T_1M_1} - \frac{K_{11}+K_{33}}{T_1M_1^2}\frac{dM_1}{dk} - \frac{K'_{22}}{T_2M_1} - \frac{K_{22}}{T_2M_1^2}\frac{dM_1}{dk} + p_y^2\left[\frac{(K_{21}^2+K_{32}^2)}{T_1T_2M_1^3}\frac{dM_1}{dk} - \frac{(K_{21}K'_{21}+K_{32}K'_{32})}{T_1T_2M_1^2}\right]$$

or simplified as

$$\frac{da}{dk} = -\frac{(K'_{11}+K'_{33})}{T_1M_1} - 2p_y^2\frac{(K_{21}K'_{21}+K_{32}K'_{32})}{T_1T_2M_1^2} + \left[p_y^2\frac{(K_{21}^2+K_{32}^2)}{T_1T_2M_1^3} - \frac{a}{M_1}\right]\frac{dM_1}{dk} \tag{4.40b}$$

then in b

$$b \equiv \frac{K_{11}K_{22}+K_{22}K_{33}}{T_1T_2M_1^2} + \frac{K_{11}K_{33}-K_{31}^2}{T_1^2M_1^2} + p_y^2\left[\frac{K_{32}^2K_{11}+K_{21}^2K_{33}-2K_{31}K_{21}K_{32}}{T_1^2T_2M_1^3}\right] \tag{4.41a}$$

$$\begin{aligned}\frac{db}{dk} = {} & \frac{K_{22}K'_{11}+K_{11}K'_{22}+K_{33}K'_{22}+K_{22}K'_{33}}{T_1T_2M_1^2} - 2\frac{(K_{11}K_{22}+K_{22}K_{33})}{T_1T_2M_1^3}\frac{dM_1}{dk} \\ & + \frac{K_{33}K'_{11}+K_{11}K'_{33}-2K_{31}K'_{31}}{T_1^2M_1^2} - 2\frac{(K_{11}K_{33}-K_{31}^2)}{T_1^2M_1^3}\frac{dM_1}{dk} \\ & + p_y^2\frac{2K_{32}K_{11}K'_{32}+K_{32}^2K'_{11}+2K_{21}K_{33}K'_{21}+K_{21}^2K'_{33}}{T_1^2T_2M_1^3} \\ & - 2p_y^2\frac{(K_{21}K_{32}K'_{31}+K_{31}K_{32}K'_{21}+K_{31}K_{21}K'_{32})}{T_1^2T_2M_1^3} \\ & - 3p_y^2\frac{(K_{32}^2K_{11}+K_{21}^2K_{33}-2K_{31}K_{21}K_{32})}{T_1^2T_2M_1^4}\frac{dM_1}{dk}\end{aligned}$$

or

$$\begin{aligned}\frac{db}{dk} = {} & \frac{K_{22}K'_{11}+K_{11}K'_{22}+K_{33}K'_{22}+K_{22}K'_{33}}{T_1T_2M_1^2} + \frac{K_{33}K'_{11}+K_{11}K'_{33}-2K_{31}K'_{31}}{T_1^2M_1^2} \\ & + p_y^2\Big[\frac{2K_{32}K_{11}K'_{32}+K_{32}^2K'_{11}+2K_{21}K_{33}K'_{21}+K_{21}^2K'_{33}}{T_1^2T_2M_1^3} \\ & - 2\frac{(K_{21}K_{32}K'_{31}+K_{31}K_{32}K'_{21}+K_{31}K_{21}K'_{32})}{T_1^2T_2M_1^3}\Big] \\ & - \left[\frac{2b}{M_1} + p_y^2\frac{(K_{32}^2K_{11}+K_{21}^2K_{33}-2K_{31}K_{21}K_{32})}{T_1^2T_2M_1^4}\right]\frac{dM_1}{dk}\end{aligned} \tag{4.41b}$$

and in c

$$c \equiv \frac{K_{31}^2K_{22}-K_{11}K_{22}K_{33}}{T_1^2T_2M_1^3} \tag{4.42a}$$

$$\frac{dc}{dk} = \frac{2K_{31}K_{22}K'_{31} + K^2_{31}K'_{22} - (K_{22}K_{33}K'_{11} + K_{11}K_{33}K'_{22} + K_{11}K_{22}K'_{33})}{T_1^2 T_2 M_1^3} - 3\frac{(K^2_{31}K_{22} - K_{11}K_{22}K_{33})}{T_1^2 T_2 M_1^4}\frac{dM_1}{dk}$$

or

$$\frac{dc}{dk} = \frac{2K_{31}K_{22}K'_{31} + K^2_{31}K'_{22} - (K_{22}K_{33}K'_{11} + K_{11}K_{33}K'_{22} + K_{11}K_{22}K'_{33})}{T_1^2 T_2 M_1^3} - \frac{3c}{M_1}\frac{dM_1}{dk} \tag{4.42b}$$

4.1.4 Special Case

As a special case of (4.23) we consider now where $\phi = \psi = 0$ are zero. We reduce $T_1 = T_2 = -1$ and lose our p_y terms, allowing us to write (4.23) as

$$\begin{bmatrix} \frac{\omega^2\Delta^2}{\alpha^2}M_1 + K_{11} & 0 & K_{31} \\ 0 & \frac{\omega^2\Delta^2}{\alpha^2}M_1 + K_{22} & 0 \\ K_{31} & 0 & \frac{\omega^2\Delta^2}{\alpha^2}M_1 + K_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = 0 \tag{4.43}$$

Expanding the determinant we find that

$$\left(\frac{\omega^2\Delta^2}{\alpha^2}M_1 + K_{22}\right)\left[\left(\frac{\omega^2\Delta^2}{\alpha^2}M_1 + K_{11}\right)\left(\frac{\omega^2\Delta^2}{\alpha^2}M_1 + K_{33}\right) - K^2_{31}\right] = 0 \tag{4.44}$$

Where we realize that M_1 and K_{22} are equivalent to the forms given by Jo et al (1996) and K_{11}, K_{22}, and K_{31} are the corrected terms for Stekl and Pratt (1998), thus we have decoupled our system into a purely acoustic case with shear wave speed and a 2D elastic case.

If one then solves (4.44) where $x = \frac{\omega\Delta}{\alpha}$, and matches up with the values calculated by the cubic equation, we identify the roots as

$$x_1 = \frac{-\frac{K_{11}+K_{33}}{M_1} + \sqrt{\frac{(K_{11}+K_{33})^2}{M_1^2} + \frac{K_{11}K_{33}-K^2_{31}}{M_1}}}{2} \tag{4.45a}$$

$$x_2 = \frac{-\frac{K_{11}+K_{33}}{M_1} - \sqrt{\frac{(K_{11}+K_{33})^2}{M_1^2} + \frac{K_{11}K_{33}-K^2_{31}}{M_1}}}{2} \tag{4.45b}$$

$$x_3 = -\frac{K_{22}}{M_1} \tag{4.45c}$$

where x_1 and x_2 are the P and SV coupled 2D elastic case from the quadratic, and x_3 is the SH acoustic case.

4.1.5 The Cubic Equation

The following is from the discussion Press et al. (1992), or for convenience P92. For the cubic $x^3+ax^2+bx+c=0$ with real coefficients (where we have defined $x=\omega\Delta$). Let

$$Q=\frac{a^2-3b}{9};\qquad R=\frac{2a^3-9ab+27c}{54} \tag{4.46}$$

with derivatives given by

$$\frac{dQ}{dk}=\frac{2a\frac{da}{dk}-3\frac{db}{dk}}{9} \tag{4.47}$$

and

$$\frac{dR}{dk}=\frac{6a^2\frac{da}{dk}-9\left(b\frac{da}{dk}+a\frac{db}{dk}\right)+27\frac{dc}{dk}}{54} \tag{4.48}$$

We first note that Q and R in our case will always be real. We can then define a discriminant $D\equiv R^2-Q^3$. If $D<0$ or as expressed in P92 $R^2<Q^3$ then the cubic equation has three real roots found by computing

4.1.6 $D<0$: Real Case

$$\theta=\cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right) \tag{4.49}$$

with derivative

$$\frac{d\theta}{dk}=-\frac{\frac{1}{\sqrt{Q^3}}\frac{dR}{dk}-\frac{3R}{2\sqrt{Q^5}}\frac{dQ}{dk}}{\frac{\sqrt{Q^3-R^2}}{Q^{3/2}}}$$

which simplifies to

$$\frac{d\theta}{dk}=\frac{-\frac{dR}{dk}+\frac{3R}{2Q}\frac{dQ}{dk}}{\sqrt{Q^3-R^2}} \tag{4.50}$$

Next the three real roots are given by

$$x_1=-2\sqrt{Q}\cos\left(\frac{\theta}{3}\right)-\frac{a}{3} \tag{4.51}$$

$$x_2=2\sqrt{Q}\cos\left(\frac{\theta+\pi}{3}\right)-\frac{a}{3} \tag{4.52}$$

$$x_3=2\sqrt{Q}\cos\left(\frac{\theta-\pi}{3}\right)-\frac{a}{3} \tag{4.53}$$

with derivatives

$$\frac{dx_1}{dk} = -\frac{1}{\sqrt{Q}}\cos\left(\frac{\theta}{3}\right)\frac{dQ}{dk} + \frac{2\sqrt{Q}}{3}\sin\left(\frac{\theta}{3}\right)\frac{d\theta}{dk} - \frac{1}{3}\frac{da}{dk} \tag{4.54}$$

$$\frac{dx_2}{dk} = \frac{1}{\sqrt{Q}}\cos\left(\frac{\theta+\pi}{3}\right)\frac{dQ}{dk} - \frac{2\sqrt{Q}}{3}\sin\left(\frac{\theta+\pi}{3}\right)\frac{d\theta}{dk} - \frac{1}{3}\frac{da}{dk} \tag{4.55}$$

$$\frac{dx_3}{dk} = \frac{1}{\sqrt{Q}}\cos\left(\frac{\theta-\pi}{3}\right)\frac{dQ}{dk} - \frac{2\sqrt{Q}}{3}\sin\left(\frac{\theta-\pi}{3}\right)\frac{d\theta}{dk} - \frac{1}{3}\frac{da}{dk} \tag{4.56}$$

Notice in this case that the roots do not appear to be ordered, however, as will be shown later they are directly related to the complex roots. It may also be worth pointing out these formulas differ from the formulas prescribed by Numerical Recipes by a minus sign and a factor of π in the argument of the cosine functions. This was the source of many headaches.

4.1.7 $D > 0$: Complex Case

Since we have real coefficients (5.6.13) and (5.6.14) of P92 are reduced to

$$A = -\text{sign}(R)\left[|R| + \sqrt{R^2 - Q^3}\right]^{1/3} \tag{4.57}$$

where we realize that the $R^2 - Q^3$ is now a real contribution by our assumption on D. Related to this is

$$B = \begin{cases} \frac{Q}{A} & A \neq 0 \\ 0 & A = 0 \end{cases} \tag{4.58}$$

with derivatives

$$\frac{dA}{dk} = -\frac{\text{sign(R)}}{3}\left(\text{sign}(R)\frac{dR}{dk} + \frac{R\frac{dR}{dk} - \frac{3Q^2}{2}\frac{dQ}{dk}}{\sqrt{R^2 - Q^3}}\right)\left(\frac{1}{|R| + \sqrt{R^2 - Q^3}}\right)^{2/3} \tag{4.59}$$

$$\frac{dB}{dk} = \frac{1}{A}\frac{dQ}{dk} - \frac{Q}{A^2}\frac{dA}{dk} \tag{4.60}$$

I have not investigated whether or not B is smooth enough such that its derivative exists for $A = 0$, however, if $A = 0$, then $\frac{dA}{dk}$ does not exist then the algorithm will fail, so it does not make any difference.

Finally we obtain our one real and two complex roots with

$$x_1 = (A+B) - \frac{a}{3} \tag{4.61a}$$

$$x_2 = -\frac{1}{2}(A+B) - \frac{a}{3} + i\frac{\sqrt{3}}{2}(A-B) \tag{4.61b}$$

$$x_3 = -\frac{1}{2}(A+B) - \frac{a}{3} - i\frac{\sqrt{3}}{2}(A-B) \tag{4.61c}$$

Our intuition dictates that x_1 will be the real P wave component, while x_2 and x_3 are the S wave components. However, we must note we are interested in magnitudes when classifying roots, and as we will find later, our intuition fails.

Lastly, the derivatives are given by

$$\frac{dx_1}{dk} = \left(\frac{dA}{dk} + \frac{dB}{dk}\right) - \frac{1}{3}\frac{da}{dk} \tag{4.62a}$$

$$\frac{dx_2}{dk} = -\frac{1}{2}\left(\frac{dA}{dk} + \frac{dB}{dk}\right) - \frac{1}{3}\frac{da}{dk} + \frac{i\sqrt{3}}{2}\left(\frac{dA}{dk} - \frac{dB}{dk}\right) \tag{4.62b}$$

$$\frac{dx_3}{dk} = -\frac{1}{2}\left(\frac{dA}{dk} + \frac{dB}{dk}\right) - \frac{1}{3}\frac{da}{dk} + \frac{i\sqrt{3}}{2}\left(\frac{dA}{dk} - \frac{dB}{dk}\right) \tag{4.62c}$$

4.1.8 Group and Phase Velocities

Now we note for both cases that $x_i = (\omega_i \Delta)^2$ hence our phase velocities, $v_{ph} = \frac{\omega}{k}$, is given by

$$v_{ph} = \frac{\omega_i}{k} = \frac{\sqrt{x_i}}{k\Delta}$$

and then applying $\omega_i \Delta = \sqrt{x_i}$ we find the derivative

$$v_{gr} = \frac{d\omega_i}{dk} = \frac{1}{2\sqrt{x_i}\Delta}\Delta\frac{dx_i}{dk} = \frac{1}{2\sqrt{x_i}}\frac{dx_i}{dk}$$

for $i = 1, 2, 3$. It is quite fortuitous that the Δ comes all the way out of the derivative of x_i to cancel with the other Δ from the $\omega\Delta$ contribution. Then dividing by the P wave speed, α, we finish with our error forms given by

$$E_{ph}^P = \frac{v_{ph}^P}{\alpha} = \frac{\sqrt{x_P}}{\alpha k \Delta} \tag{4.63a}$$

$$E_{ph}^{SV} = \frac{v_{ph}^{SV}}{\beta} = \frac{v_{ph}^{SV}}{R\,\alpha} = \frac{\sqrt{x_{SV}}}{R\,\alpha k\Delta} \tag{4.63b}$$

$$E_{ph}^{SH} = \frac{v_{ph}^{SH}}{\beta} = \frac{v_{ph}^{SH}}{R\,\alpha} = \frac{\sqrt{x_{SH}}}{R\,\alpha k\Delta} \tag{4.63c}$$

$$E_{gr}^{P} = \frac{v_{gr}^{P}}{\alpha} = \frac{1}{2\alpha\sqrt{x_P}}\frac{dx_P}{dk} \tag{4.63d}$$

$$E_{gr}^{SV} = \frac{v_{gr}^{SV}}{\beta} = \frac{v_{gr}^{SV}}{R\,\alpha} = \frac{1}{2R\,\alpha\sqrt{x_{SV}}}\frac{dx_{SV}}{dk} \tag{4.63e}$$

$$E_{gr}^{SH} = \frac{v_{gr}^{SH}}{\beta} = \frac{1}{2R\,\alpha\sqrt{x_{SH}}}\frac{dx_{SH}}{dk} \tag{4.63f}$$

4.1.9 Derivation of Real Roots

Consider the roots of the cubic given by

$$x_1 = -\frac{a}{3} - \frac{1}{3}\sqrt[3]{R + \sqrt{R^2 - Q^3}} \tag{4.64a}$$

$$x_2 = -\frac{a}{3} + \frac{1+i\sqrt{3}}{3}\sqrt[3]{R + \sqrt{R^2 - Q^3}} + \frac{1-i\sqrt{3}}{3}\sqrt[3]{R - \sqrt{R^2 - Q^3}} \tag{4.64b}$$

$$x_2 = -\frac{a}{3} + \frac{1-i\sqrt{3}}{3}\sqrt[3]{R + \sqrt{R^2 - Q^3}} + \frac{1+i\sqrt{3}}{3}\sqrt[3]{R - \sqrt{R^2 - Q^3}} \tag{4.64c}$$

We will use the assumption that $R^2 - Q^3 < 0$. By definition the argument under the third root is complex, thus

$$-\frac{a}{3} - \frac{1}{3}\sqrt[3]{R + \sqrt{R^2 - Q^3}} = -\frac{a}{3} - \frac{1}{3}\sqrt[3]{x + iy} = -\frac{1}{3}\rho^{1/3}e^{i\theta}$$

$$\begin{aligned}
&= -\frac{a}{3} + \frac{1+i\sqrt{3}}{3}\sqrt[3]{R + \sqrt{R^2 - Q^3}} + \frac{1-i\sqrt{3}}{3}\sqrt[3]{R - \sqrt{R^2 - Q^3}} \\
&= -\frac{a}{3} + \rho^{1/3}\left(e^{i\pi/3}e^{i\theta/3} + e^{-i\theta/3}e^{-i\pi/3}\right) \\
&= -\frac{a}{3} + 2\rho^{1/3}\cos\left(\frac{\theta+\pi}{3}\right)
\end{aligned}$$

and similarly in x_3 we get

$$x_3 = -\frac{a}{3} + 2\rho^{1/3}\cos\left(\frac{\theta - \pi}{3}\right)$$

where $x = R$ and $y = \sqrt{Q^3 - R^2}$ hence

$$\rho = (x + iy)(x - iy) = \sqrt{R^2 + Q^3 - R^2} = \sqrt{Q^3}$$

and $\theta = \tan^{-1}\left(\frac{\sqrt{Q^3-R^2}}{R}\right)$. Then remembering our triangle trig we have a hypotenuse of $\sqrt{Q^3}$, therefore $\theta = \cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right)$. Now there is a relationship between the real and complex roots. Numerical tests indicate that $x_1 \Rightarrow SH$, $x_2 \Rightarrow SV$, and $x_3 \Rightarrow P$ with $\theta \approx \pi$. Given this we can imagine x_3 lying around 0°, x_2 around 120°, and x_1 around 60°.

4.1.10 Numerical Integration

To approximate the integral in (4.4) we will use a 2D Simpson's rule given by Mathews and Fink (2004). That is

$$\int\int_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx \approx \text{S2D}(f,h,k) \tag{4.65}$$

where

$$\begin{aligned}
\tfrac{9}{hk}\text{S2D} = \;& f(a,c) + f(a,d) + f(b,c) + f(b,d) \\
& +4\sum_{j=1}^{n} f(a,y_{2j-1}) + 2\sum_{j=1}^{n-1} f(a,y_{2j}) + 4\sum_{j=1}^{n} f(b,y_{2j-1}) + 2\sum_{j=1}^{n-1} f(b,y_{2j}) \\
& +4\sum_{i=1}^{m} f(x_{2i-1},c) + 2\sum_{i=1}^{m-1} f(x_{2i},c) + 4\sum_{i=1}^{m} f(x_{2i-1},d) + 2\sum_{i=1}^{m-1} f(x_{2i},d) \\
& +16\sum_{j=1}^{n}\left(\sum_{i=1}^{m} f(x_{2i-1},y_{2j-1})\right) + 8\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m} f(x_{2i-1},y_{2j})\right) \\
& +8\sum_{j=1}^{n}\left(\sum_{i=1}^{m-1} f(x_{2i},y_{2j-1})\right) + 4\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m-1} f(x_{2i},y_{2j})\right)
\end{aligned} \tag{4.66}$$

over the rectangle, $R = \{(x, y) : a \leq x \leq b,\, c \leq y \leq d\}$ where x is divided into $2m$ subintervals by $h = \frac{b-a}{2m}$, $i = 0, 1, \cdots, 2m$ and y is divided into $2n$ subintervals by $k = \frac{d-c}{2n}$, $j = 0, 1, \cdots, 2n$. Notice this definition of k is slightly different than the one from Mathews, mainly because he made a mistake and prescribed k over $\frac{b-a}{2n}$. His examples still work because he only uses the region $[0, 1]$ for limits on both integrals. The reasons Simpson's rule was utilized for numerical integration is first for it's ease of implementation and that the error here can be shown to be of $\mathbf{O}(h^4) + \mathbf{O}(k^4)$, thereby dominating a simple trapezoidal or box integration. Other schemes such as Gauss family integraiton rules, which in general will converge much faster assuming a 'smooth' integral can be implemented, however they are a little more difficult. An excellent introduction to generating Gauss family integrals can be found in Golub and Welsch (1962).

CHAPTER 5
Discussion and Conclusions

As a demonstration of the proposed method we provide a numerical example of the forward problem. Here we considered a homogeneous background of compressional wavespeed of $6,000$ meters per second with a faster compressional wave speed of $7,000$. The corresponding shear wave speeds are approximately $3,500$ meters per second and $4,000$ meters per second, figure 5.1. The model considers angle of incidence of 20° and back azimuth of 10°. The effective mantle velocity is $6,000$ meters per second, resulting in a $p_y = 0.0098$ kilometers per second. There were 420 grid points in x and 104 grid points in z spaced at 400 meters which equates to a length of 168 kilometers length and 41 kilometers depth. The band consists of 80 frequencies ranging from .01 Hz to .8 Hz. The source time function was a Ricker wavelet. The time window is 1 minute. The run was serial on a 4 processor Linux dual core AMD Opteron Processor and completed in approximately one hour, for $3 \times 420 \times 104 = 131,040$ degrees of freedom.

A snapshot of the system response also is given at 11.2 seconds in figure 5.1. All components are scaled equally, and for plotting reasons, are given a value between 0 and 255. On the right side of the model we see the energy is predominately in the vertical, w, component. As we get closer to the box, the effect of the velocity contrast resulting in a stronger u and v response. Finally, on the left, the u and v components show a strong S conversion resulting from the free surface. There is also some u and v energy ahead of this phase which corresponds to the P reflection, which is most noticeable in the w response.

To summarize, we have considered the $3D$ elastic problem, imposed a geologically valid assumption of homogeneity in one direction thus simplifying to 2.5 dimensions. To aid in numerical accuracy we then expressed the corresponding equations of motion in a rotated frame, which is related by a mass lumping and weighting scheme, then proceeded to discretize, via finite differences. Next, for accuracy modeling of plane waves, we introduced the scattered field total field approach. Then,

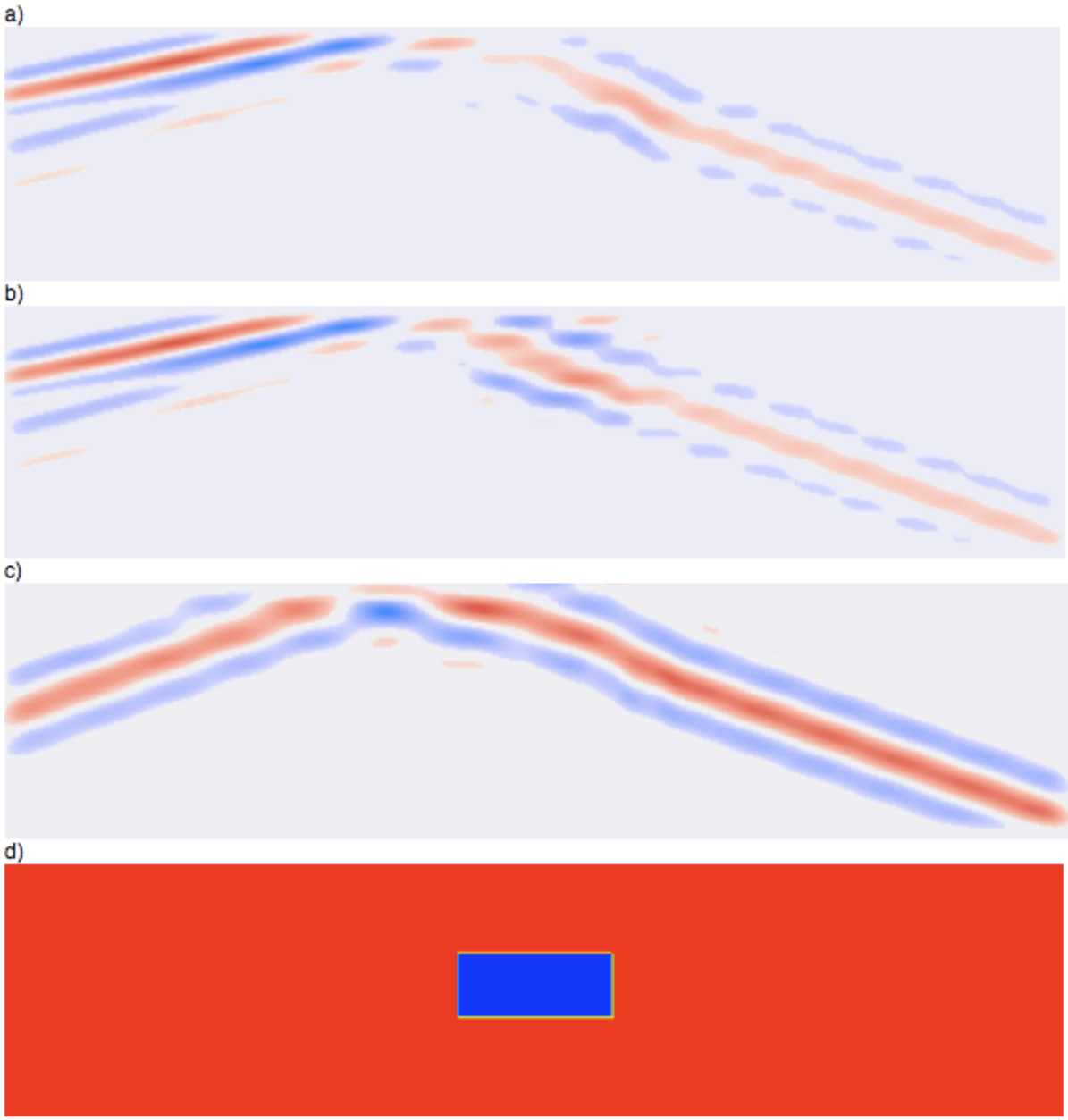

Figure 5.1: (a), (b), and (c) are the u, v and w response, respectively, to a velocity model given by (d) at 11.2 seconds. The background layer is the compressional velocity with the red at $6,000$ meters per second, and the blue at $7,000$ meters per second. The model physically spans 168 kilometers in length and 41 kilometers in depth.

we developed an analytical technique to produce a solution not just at the free surface, but over the entire model, based on the Haskell propagator approach. After, we developed a method for reducing numerical anisotropy and dispersion. Finally, we presented a small example to demonstrate the numerical efficacy of the method.

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APPENDIX A
Derivation of the Finite Difference Forms

A.1 Finite Difference Forms in u

In this appendix we find the discretized form for (1.13a), given by

$$(\mu p_y^2-\rho)\omega^2 u = R_{2x}(\lambda+2\mu)\frac{\partial u}{\partial x}+R_{1x}^2\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x}\right]+R_{1x}R_{1z}\left(\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right]+\frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]\right)$$
$$+R_{2z}\mu\frac{\partial u}{\partial z}+R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial u}{\partial z}\right]-R_{1x}i\omega p_y\left(\frac{\partial}{\partial x}\left[\lambda v\right]+\mu\frac{\partial v}{\partial x}\right)$$

1u.

$$(\rho-p_y^2\mu)\omega^2 u=\rho'\omega^2 u \quad \Rightarrow b\omega^2\rho' u_{i,k}+\omega^2\rho'\frac{(1-b)}{4}\left(u_{i+1,k}+u_{i-1,k}+u_{i,k+1}+u_{i,k-1}\right)$$

2u.

$$R_{1x}^2\frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x}\right]\Rightarrow_{F1} \quad \frac{R_{1x}^2}{\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k}\left(u_{i+1,k}-u_{i,k}\right)-(\lambda+2\mu)_{i-1/2\,,k}\left(u_{i,k}-u_{i-1,k}\right)\right]$$

3u.

$$R_{1x}R_{1z}\frac{\partial}{\partial x}\left[\lambda\frac{\partial w}{\partial z}\right]\Rightarrow_{F2} \quad \frac{R_{1x}R_{1z}}{4\Delta^2}\left[\lambda_{i+1,k}\left(w_{i+1,k+1}-w_{i+1,k-1}\right)-\lambda_{i-1,k}\left(w_{i-1,k+1}-w_{i-1,k-1}\right)\right]$$

4u.

$$R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial u}{\partial z}\right]\Rightarrow_{F3} \quad \frac{R_{1z}^2}{\Delta^2}\left[\mu_{i,k+1/2}\left(u_{i,k+1}-u_{i,k}\right)-\mu_{i,k-1/2}\left(u_{i,k}-u_{i,k-1}\right)\right]$$

5u.

$$R_{1x}R_{1z}\frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]\Rightarrow_{F4} \quad \frac{R_{1x}R_{1z}}{4\Delta^2}\left[\mu_{i,k+1}\left(w_{i+1,k+1}-w_{i-1,k+1}\right)-\mu_{i,k-1}\left(w_{i+1,k-1}-w_{i-1,k-1}\right)\right]$$

6u.

$$-R_{1x}i\omega p_y\frac{\partial}{\partial x}[\lambda v] \Rightarrow_{F5} \quad -R_{1x}\frac{i\omega p_y}{2\Delta}\left[\lambda_{i+1/2\,,k}\left(v_{i+1,k}+v_{i,k}\right)-\lambda_{i-1/2\,,k}\left(v_{i,k}+v_{i-1,k}\right)\right]$$

7u.

$$-R_{1x}i\omega p_y\mu\frac{\partial v}{\partial x} \Rightarrow_{F6} \quad -R_{1x}\frac{i\omega p_y}{2\Delta}\left[\mu_{i+1/2\,,k}\left(v_{i+1,k}-v_{i,k}\right)+\mu_{i-1/2\,,k}\left(v_{i,k}-v_{i-1,k}\right)\right]$$

New terms: 8u.

$$R_{2z}\mu\frac{\partial u}{\partial z} \Rightarrow_{F7} \quad \frac{R_{2z}}{2\Delta}\left[\mu_{i\,,k+1/2}\left(u_{i,k+1}-u_{i,k}\right)+\mu_{i\,,k-1/2}\left(u_{i,k}-u_{i,k-1}\right)\right]$$

9u.

$$R_{2x}(\lambda{+}2\mu)\frac{\partial u}{\partial x} \Rightarrow_{F6} \quad \frac{R_{2x}}{2\Delta}\left[(\lambda+2\mu)_{i+1/2\,,k}\left(u_{i+1,k}-u_{i,k}\right)+(\lambda+2\mu)_{i-1/2\,,k}\left(u_{i,k}-u_{i-1,k}\right)\right]$$

The rotated terms are from (1.15a)

$$\begin{aligned}(\mu p_y^2{-}\rho)\omega^2 u = {}& \frac{R_{2x}}{\sqrt{2}}(\lambda{+}2\mu)\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)+\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)(\lambda{+}2\mu)\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\\ &+\frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\lambda\left(\frac{\partial w}{\partial x'}+\frac{\partial w}{\partial z'}\right)\right]+\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right]\right\}\\ &+\frac{R_{2z}}{\sqrt{2}}\left\{\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right\}+\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\mu\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right]\\ &-R_{1x}i\frac{\omega p_y}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\lambda v]-\frac{\partial}{\partial z'}[\lambda v]+\mu\left(\frac{\partial v}{\partial x'}-\frac{\partial v}{\partial z'}\right)\right\}\end{aligned}$$

1u-R.

$$\begin{aligned}&\frac{R_{2x}}{\sqrt{2}}\left[(\lambda+2\mu)\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right] \Rightarrow_{RF1-RF2}\\ &=\frac{R_{2x}}{4\Delta}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}-u_{i,k}\right)+(\lambda+2\mu)_{i-1/2\,,k-1/2}\left(u_{i,k}-u_{i-1,k-1}\right)\right]\\ &-\frac{R_{2x}}{4\Delta}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}-u_{i,k}\right)+(\lambda+2\mu)_{i+1/2\,,k-1/2}\left(u_{i,k}-u_{i+1,k-1}\right)\right]\end{aligned}$$

2u-R.

$$\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[(\lambda+2\mu)\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right]=$$
$$\frac{R_{1x}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial u}{\partial x'}\right]-\frac{R_{1x}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial u}{\partial z'}\right]-\frac{R_{1x}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial u}{\partial x'}\right]+\frac{R_{1x}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial u}{\partial z'}\right]$$

$$(i)\qquad \frac{R_{1x}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial u}{\partial x'}\right]\Rightarrow_{RF3}$$
$$\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}-u_{i,k}\right)-(\lambda+2\mu)_{i-1/2\,,k-1/2}\left(u_{i,k}-u_{i-1,k-1}\right)\right]$$

$$(ii)\qquad -\frac{R_{1x}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial u}{\partial z'}\right]\Rightarrow_{RF4}$$
$$-\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\left(u_{i,k+1}-u_{i+1,k}\right)-(\lambda+2\mu)_{i-1/2\,,k-1/2}\left(u_{i-1,k}-u_{i,k-1}\right)\right]$$

$$(iii)\qquad -\frac{R_{1x}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial u}{\partial x'}\right]\Rightarrow_{RF5}$$
$$-\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2}\left(u_{i,k+1}-u_{i-1,k}\right)-(\lambda+2\mu)_{i+1/2\,,k-1/2}\left(u_{i+1,k}-u_{i,k-1}\right)\right]$$

$$(iv)\frac{R_{1x}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial u}{\partial z'}\right]\Rightarrow_{RF6}$$
$$\frac{R_{1x}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}-u_{i,k}\right)-(\lambda+2\mu)_{i+1/2\,,k-1/2}\left(u_{i,k}-u_{i+1,k-1}\right)\right]$$

3u-R.

$$\frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\lambda\left(\frac{\partial w}{\partial x'}+\frac{\partial w}{\partial z'}\right)\right]+\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right]\right\}$$
$$=\frac{R_{1x}R_{1z}}{2}\left(\frac{\partial}{\partial x'}\left[\lambda\frac{\partial w}{\partial z'}\right]+\frac{\partial}{\partial x'}\left[\lambda\frac{\partial w}{\partial x'}\right]-\frac{\partial}{\partial z'}\left[\lambda\frac{\partial w}{\partial z'}\right]-\frac{\partial}{\partial z'}\left[\lambda\frac{\partial w}{\partial x'}\right]\right)$$
$$+\frac{R_{1x}R_{1z}}{2}\left(\frac{\partial}{\partial z'}\left[\mu\frac{\partial w}{\partial x'}\right]+\frac{\partial}{\partial x'}\left[\mu\frac{\partial w}{\partial x'}\right]-\frac{\partial}{\partial x'}\left[\mu\frac{\partial w}{\partial z'}\right]-\frac{\partial}{\partial z'}\left[\mu\frac{\partial w}{\partial z'}\right]\right)$$
$$=\frac{R_{1x}R_{1z}}{2}\left(\frac{\partial}{\partial x'}\left[(\lambda-\mu)\frac{\partial w}{\partial z'}\right]+\frac{\partial}{\partial x'}\left[(\lambda+\mu)\frac{\partial w}{\partial x'}\right]-\frac{\partial}{\partial z'}\left[(\lambda+\mu)\frac{\partial w}{\partial z'}\right]-\frac{\partial}{\partial z'}\left[(\lambda-\mu)\frac{\partial w}{\partial x'}\right]\right)$$

$$(i)\qquad\frac{R_{1x}R_{1z}}{2}\frac{\partial}{\partial x'}\left[(\lambda-\mu)\frac{\partial w}{\partial z'}\right]\Rightarrow_{RF4}$$
$$\frac{R_{1x}R_{2x}}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k+1/2}\left(w_{i,k+1}-w_{i+1,k}\right)-(\lambda-\mu)_{i-1/2\,,k-1/2}\left(w_{i-1,k}-w_{i,k-1}\right)\right]$$

$$(ii)\qquad\frac{R_{1x}R_{1z}}{2}\frac{\partial}{\partial x'}\left[(\lambda+\mu)\frac{\partial w}{\partial x'}\right]\Rightarrow_{RF3}$$
$$=\frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k+1/2}\left(w_{i+1,k+1}-w_{i,k}\right)-(\lambda+\mu)_{i-1/2\,,k-1/2}\left(w_{i,k}-w_{i-1,k-1}\right)\right]$$

$$(iii)\qquad-\frac{R_{1x}R_{1z}}{2}\frac{\partial}{\partial z'}\left[(\lambda+\mu)\frac{\partial w}{\partial z'}\right]\Rightarrow_{RF6}$$
$$-\frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda+\mu)_{i-1/2\,,k+1/2}\left(w_{i-1,k+1}-w_{i,k}\right)-(\lambda+\mu)_{i+1/2\,,k-1/2}\left(w_{i,k}-w_{i+1,k-1}\right)\right]$$

$$(iv)\qquad-\frac{R_{1x}R_{1z}}{2}\frac{\partial}{\partial z'}\left[(\lambda-\mu)\frac{\partial w}{\partial x'}\right]\Rightarrow_{RF5}$$
$$-\frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k+1/2}\left(w_{i,k+1}-w_{i-1,k}\right)-(\lambda-\mu)_{i+1/2\,,k-1/2}\left(w_{i+1,k}-w_{i,k-1}\right)\right]$$

4u-R.

$$\frac{R_{2x}}{\sqrt{2}}\left[\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right]\Rightarrow_{RF2+RF1}$$
$$\frac{R_{2x}}{4\Delta}\left[\mu_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}-u_{i,k}\right)+\mu_{i+1/2\,,k-1/2}\left(u_{i,k}-u_{i+1,k-1}\right)\right]$$
$$+\frac{R_{2x}}{4\Delta}\left[\mu_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}-u_{i,k}\right)+\mu_{i-1/2\,,k-1/2}\left(u_{i,k}-u_{i-1,k-1}\right)\right]$$

5u-R.

$$\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right]$$
$$=\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}\left[\mu\frac{\partial u}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial u}{\partial x'}\right]+\frac{\partial}{\partial x'}\left[\mu\frac{\partial u}{\partial z'}\right]+\frac{\partial}{\partial x'}\left[\mu\frac{\partial u}{\partial x'}\right]\right)$$

(i)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial z'}\left[\mu\frac{\partial u}{\partial z'}\right]\Rightarrow_{RF6}\qquad\frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}-u_{i,k}\right)-\mu_{i+1/2\,,k-1/2}\left(u_{i,k}-u_{i+1,k-1}\right)\right]$$

(ii)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial z'}\left[\mu\frac{\partial u}{\partial x'}\right]\Rightarrow_{RF5}\qquad\frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}\left(u_{i,k+1}-u_{i-1,k}\right)-\mu_{i+1/2\,,k-1/2}\left(u_{i+1,k}-u_{i,k-1}\right)\right]$$

(iii)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial x'}\left[\mu\frac{\partial u}{\partial z'}\right]\Rightarrow_{RF4}\qquad\frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2}\left(u_{i,k+1}-u_{i+1,k}\right)-\mu_{i-1/2\,,k-1/2}\left(u_{i-1,k}-u_{i,k-1}\right)\right]$$

(iv)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial x'}\left[\mu\frac{\partial u}{\partial x'}\right]\Rightarrow_{RF3}\qquad\frac{R_{1z}^2}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}-u_{i,k}\right)-\mu_{i-1/2\,,k-1/2}\left(u_{i,k}-u_{i-1,k-1}\right)\right]$$

6u-R.

$$
\begin{aligned}
&- R_{1x}\frac{i\omega p_y}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\lambda v]-\frac{\partial}{\partial z'}[\lambda v]+\mu\left(\frac{\partial v}{\partial x'}-\frac{\partial v}{\partial z'}\right)\right\}\Rightarrow_{RF7-RF8+RF1-RF2}\\
&\quad- R_{1x}\frac{i\omega p_y}{4\Delta}\left[\lambda_{i+1/2\,,k+1/2}\left(v_{i+1,k+1}+v_{i,k}\right)-\lambda_{i-1/2\,,k-1/2}\left(v_{i,k}+v_{i-1,k-1}\right)\right]\\
&\quad+ R_{1x}\frac{i\omega p_y}{4\Delta}\left[\lambda_{i-1/2\,,k+1/2}\left(v_{i-1,k+1}+v_{i,k}\right)-\lambda_{i+1/2\,,k-1/2}\left(v_{i,k}-v_{i+1,k-1}\right)\right]\\
&\quad- R_{1x}\frac{i\omega p_y}{4\Delta}\left[\mu_{i+1/2\,,k+1/2}\left(v_{i+1,k+1}-v_{i,k}\right)+\mu_{i-1/2\,,k-1/2}\left(v_{i,k}-v_{i-1,k-1}\right)\right]\\
&\qquad+ R_{1x}\frac{i\omega p_y}{4\Delta}\left[\mu_{i-1/2\,,k+1/2}\left(v_{i-1,k+1}-v_{i,k}\right)+\mu_{i+1/2\,,k-1/2}\left(v_{i,k}-v_{i+1,k-1}\right)\right]
\end{aligned}
$$

In deriving the inverse problem, we are also interested in the forms outside of the PML. Direct simplification of the non-rotated u terms gives

$$\frac{A5UU}{u_{i,k}}\Rightarrow b\omega^2\rho'-\frac{1}{\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k}+(\lambda+2\mu)_{i-1/2\,,k}+\mu_{i\,,k+1/2}+\mu_{i\,,k-1/2}\right] \tag{A.1a}$$

$$\frac{A8UU}{u_{i+1,k}}\Rightarrow\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}(\lambda+2\mu)_{i+1/2\,,k}$$

$$\frac{A2UU}{u_{i-1,k}}\Rightarrow\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}(\lambda+2\mu)_{i-1/2\,,k} \tag{A.1b}$$

$$\frac{A6UU}{u_{i,k+1}}\Rightarrow\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}\mu_{i\,,k+1/2} \tag{A.1c}$$

$$\frac{A4UU}{u_{i,k-1}}\Rightarrow\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}\mu_{i\,,k-1/2} \tag{A.1d}$$

$$\frac{A9UW}{w_{i+1,k+1}}\Rightarrow\frac{1}{4\Delta^2}(\lambda_{i+1,k}+\mu_{i,k+1}) \tag{A.1e}$$

$$\frac{A7UW}{w_{i+1,k-1}}\Rightarrow-\frac{1}{4\Delta^2}(\lambda_{i+1,k}+\mu_{i,k-1}) \tag{A.1f}$$

$$\frac{A3UW}{w_{i-1,k+1}}\Rightarrow-\frac{1}{4\Delta^2}(\lambda_{i-1,k}+\mu_{i,k+1}) \tag{A.1g}$$

$$\frac{A1UW}{w_{i-1,k+1}}\Rightarrow\frac{1}{4\Delta^2}(\lambda_{i-1,k}+\mu_{i,k-1}) \tag{A.1h}$$

$$\frac{A5UV}{v_{i,k}}\Rightarrow\frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i-1/2\,,k}-(\lambda-\mu)_{i+1/2\,,k}\right] \tag{A.1i}$$

$$\frac{A2UV}{v_{i-1,k}} \Rightarrow \frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i-1/2\,,k} \tag{A.1j}$$

$$\frac{A8UV}{v_{i+1,k}} \Rightarrow -\frac{i\omega p_y}{2\Delta}\left[(\lambda+\mu)_{i+1/2\,,k}\right] \tag{A.1k}$$

Likewise, the rotated out of PML terms in u are given by

$$\frac{A5UU}{u_{i,k}} \Rightarrow$$
$$-\frac{1}{4\Delta^2}\left[(\lambda+3\mu)_{i+1/2\,,k+1/2}+(\lambda+3\mu)_{i-1/2\,,k-1/2}+(\lambda+3\mu)_{i-1/2\,,k+1/2}+(\lambda+3\mu)_{i+1/2\,,k-1/2}\right] \tag{A.2a}$$

$$\frac{A9UU}{u_{i+1,k+1}} \Rightarrow \frac{1}{4\Delta^2}(\lambda+3\mu)_{i+1/2\,,k+1/2} \tag{A.2b}$$

$$\frac{A1UU}{u_{i-1,k-1}} \Rightarrow \frac{1}{4\Delta^2}(\lambda+3\mu)_{i-1/2\,,k-1/2} \tag{A.2c}$$

$$\frac{A3UU}{u_{i-1,k+1}} \Rightarrow \frac{1}{4\Delta^2}(\lambda+3\mu)_{i-1/2\,,k+1/2} \tag{A.2d}$$

$$\frac{A7UU}{u_{i+1,k-1}} \Rightarrow \frac{1}{4\Delta^2}(\lambda+3\mu)_{i+1/2\,,k-1/2} \tag{A.2e}$$

$$\frac{A6UU}{u_{i,k+1}} \Rightarrow -\frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k+1/2}+(\lambda+\mu)_{i-1/2\,,k+1/2}\right] \tag{A.2f}$$

$$\frac{A8UU}{u_{i+1,k}} \Rightarrow \frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k+1/2}+(\lambda+\mu)_{i+1/2\,,k-1/2}\right] \tag{A.2g}$$

$$\frac{A2UU}{u_{i-1,k}} \Rightarrow \frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i-1/2\,,k+1/2}+(\lambda+\mu)_{i-1/2\,,k-1/2}\right] \tag{A.2h}$$

$$\frac{A4UU}{u_{i,k-1}} \Rightarrow -\frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i-1/2\,,k-1/2}+(\lambda+\mu)_{i+1/2\,,k-1/2}\right] \tag{A.2i}$$

$$\frac{A5UW}{w_{i,k}} \Rightarrow$$
$$\frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k-1/2}+(\lambda+\mu)_{i-1/2\,,k+1/2}-(\lambda+\mu)_{i+1/2\,,k+1/2}-(\lambda+\mu)_{i-1/2\,,k-1/2}\right] \tag{A.2j}$$

$$\frac{A6UW}{w_{i,k+1}} \Rightarrow \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k+1/2}\right] \tag{A.2k}$$

$$\frac{A8UW}{w_{i+1,k}} \Rightarrow \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2}\right] \tag{A.2l}$$

$$\frac{A2UW}{w_{i-1,k}} \Rightarrow \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k-1/2}\right] \tag{A.2m}$$

$$\frac{A4UW}{w_{i,k-1}} \Rightarrow \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{A.2n}$$

$$\frac{A9UW}{w_{i+1,k+1}} \Rightarrow \frac{1}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{A.2o}$$

$$\frac{A1UW}{w_{i-1,k-1}} \Rightarrow \frac{1}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{A.2p}$$

$$\frac{A3UW}{w_{i-1,k+1}} \Rightarrow -\frac{1}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{A.2q}$$

$$\frac{A7UW}{w_{i+1,k-1}} \Rightarrow -\frac{1}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{A.2r}$$

$$\frac{A5UV}{v_{i,k}} \Rightarrow$$

$$\frac{i\omega p_y}{4\Delta}\left[(\lambda-\mu)_{i-1/2\,,k-1/2} + (\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{A.2s}$$

$$\frac{A9UV}{v_{i+1,k+1}} \Rightarrow -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{A.2t}$$

$$\frac{A1UV}{v_{i-1,k-1}} \Rightarrow \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{A.2u}$$

$$\frac{A3UV}{v_{i-1,k+1}} \Rightarrow \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{A.2v}$$

$$\frac{A7UV}{v_{i+1,k-1}} \Rightarrow -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{A.2w}$$

A.2 Finite Difference Forms in v

Recalling our general equation of motion in the non-rotated from (1.13b)

$$-(p_y^2(\lambda+2\mu)-\rho)\omega^2 v + R_{2x}\left(\mu\frac{\partial v}{\partial x}\right) + R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right] + R_{2z}\mu\frac{\partial v}{\partial z} + R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial v}{\partial z}\right] - i\omega p_y\left(R_{1x}\left\{\frac{\partial}{\partial x}\left[\mu u\right] + \lambda\frac{\partial u}{\partial x}\right\} + R_{1z}\left\{\frac{\partial}{\partial z}\left[\mu w\right] + \lambda\frac{\partial w}{\partial z}\right\}\right) = f$$

Non rotated part:

1v.

$$(\rho - p_y^2(\lambda+2\mu))\omega^2 v = \rho''\omega^2 v \Rightarrow \qquad b\omega^2\rho'' v_{i,k} + \omega^2\rho''\frac{(1-b)}{4}\left(v_{i+1,k}+v_{i-1,k}+v_{i,k+1}+v_{i,k-1}\right)$$

2v.

$$R_{2x}\left(\mu\frac{\partial v}{\partial x}\right) \Rightarrow_{F6} \qquad \frac{R_{2x}}{2\Delta}\left[\mu_{i+1/2\,,k}\left(v_{i+1,k}-v_{i,k}\right)+\mu_{i-1/2\,,k}\left(v_{i,k}-v_{i-1,k}\right)\right]$$

3v.

$$R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right] \Rightarrow_{F1} \qquad \frac{R_{1x}^2}{\Delta^2}\left[\mu_{i+1/2\,,k}\left(v_{i+1,k}-v_{i,k}\right)-\mu_{i-1/2\,,k}\left(v_{i,k}-v_{i-1,k}\right)\right]$$

4v.

$$R_{2z}\left[\mu\frac{\partial v}{\partial z}\right] \Rightarrow_{F7} \qquad \frac{R_{2z}}{2\Delta}\left[\mu_{i,k+1/2}\left(v_{i,k+1}-v_{i,k}\right)+\mu_{i,k-1/2}\left(v_{i,k}-v_{i,k-1}\right)\right]$$

5v.

$$R_{1z}^2\frac{\partial}{\partial z}\left[\mu\frac{\partial v}{\partial z}\right] \Rightarrow_{F3} \qquad \frac{R_{1z}^2}{\Delta^2}\left[\mu_{i,k+1/2}\left(v_{i,k+1}-v_{i,k}\right)-\mu_{i,k-1/2}\left(v_{i,k}-v_{i,k-1}\right)\right]$$

6v.

$$-R_{1x}i\omega p_y\left(\frac{\partial}{\partial x}[\mu u]\right) \Rightarrow_{F5} \qquad -R_{1x}\frac{i\omega p_y}{2\Delta}\left[\mu_{i+1/2\,,k}\left(u_{i+1,k}+u_{i,k}\right)-\mu_{i-1/2\,,k}\left(u_{i,k}+u_{i-1,k}\right)\right]$$

7v.

$$-R_{1x}i\omega p_y\left(\lambda\frac{\partial u}{\partial x}\right)\Rightarrow_{F6}\qquad -R_{1x}\frac{i\omega p_y}{2\Delta}\left[\lambda_{i+1/2\,,k}\left(u_{i+1,k}-u_{i,k}\right)+\lambda_{i-1/2\,,k}\left(u_{i,k}-u_{i-1,k}\right)\right]$$

8v.

$$-R_{1z}i\omega p_y\left(\frac{\partial}{\partial z}[\mu w]\right)\Rightarrow_{F8}\qquad -R_{1z}\frac{i\omega p_y}{2\Delta}\left[\mu_{i,k+1/2}(w_{i,k+1}+w_{i,k})-\mu_{i,k-1/2}(w_{i,k}+w_{i,k-1})\right]$$

9v.

$$-R_{1z}i\omega p_y\left(\lambda\frac{\partial w}{\partial z}\right)\Rightarrow_{F7}\qquad -R_{1z}\frac{i\omega p_y}{2\Delta}\left[\lambda_{i,k+1/2}\left(w_{i,k+1}-w_{i,k}\right)+\lambda_{i,k-1/2}\left(w_{i,k}-w_{i,k-1}\right)\right]$$

And for our rotated form in v given by (1.15b)

$$\begin{aligned}(p_y^2(\lambda+2\mu)-\rho)\omega^2 v &= \frac{\mu}{\sqrt{2}}\left(\frac{\partial v}{\partial x'}(R_{2x}+R_{2z})+\frac{\partial v}{\partial z'}(R_{2z}-R_{2x})\right)\\ \frac{1}{2}\left[(R_{1x}^2+R_{1z}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial x'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial z'}\right]\right)+(R_{1z}^2-R_{1x}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial x'}\right]\right)\right]&\\ -i\omega p_y\frac{R_{1x}}{\sqrt{2}}\left\{\frac{\partial}{\partial x'}[\mu u]-\frac{\partial}{\partial z'}[\mu u]+\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right\}&\\ -i\omega p_y\frac{R_{1z}}{\sqrt{2}}\left\{\frac{\partial}{\partial z'}[\mu w]+\frac{\partial}{\partial x'}[\mu w]+\lambda\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right\}&\end{aligned}$$

then we find finite difference forms

1v.

$$\begin{aligned}&\frac{\mu}{\sqrt{2}}\left[\left(\frac{\partial v}{\partial x'}(R_{2x}+R_{2z})+\frac{\partial v}{\partial z'}(R_{2z}-R_{2x})\right)\right]\Rightarrow_{RF1+RF2}\\ &\quad=\frac{(R_{2x}+R_{2z})}{4\Delta}\left[\mu_{i+1/2\,,k+1/2}\left(v_{i+1,k+1}-v_{i,k}\right)+\mu_{i-1/2\,,k-1/2}\left(v_{i,k}-v_{i-1,k-1}\right)\right]\\ &\qquad+\frac{(R_{2z}-R_{2x})}{4\Delta}\left[\mu_{i-1/2\,,k+1/2}\left(v_{i-1,k+1}-v_{i,k}\right)+\mu_{i+1/2\,,k-1/2}\left(v_{i,k}-v_{i+1,k-1}\right)\right]\end{aligned}$$

2v.

$$\frac{1}{2}\left\{(R_{1x}^2+R_{1z}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial x'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial z'}\right]\right)+(R_{1z}^2-R_{1x}^2)\left(\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial x'}\right]\right)\right\}$$

(i)

$$\frac{(R_{1x}^2+R_{1z}^2)}{2}\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial x'}\right]\Rightarrow_{RF3}$$
$$\frac{(R_{1x}^2+R_{1z}^2)}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2}\left(v_{i+1,k+1}-v_{i,k}\right)-\mu_{i-1/2\,,k-1/2}\left(v_{i,k}-v_{i-1,k-1}\right)\right]$$

(ii)

$$\frac{(R_{1x}^2-R_{1z}^2)}{2}\frac{\partial}{\partial x'}\left[\mu\frac{\partial v}{\partial z'}\right]\Rightarrow_{RF4}$$
$$\frac{(R_{1x}^2-R_{1z}^2)}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2}\left(v_{i,k+1}-v_{i+1,k}\right)-\mu_{i-1/2\,,k-1/2}\left(v_{i-1,k}-v_{i,k-1}\right)\right]$$

(iii)

$$\frac{(R_{1z}^2-R_{1x}^2)}{2}\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial x'}\right]\Rightarrow_{RF5}$$
$$\frac{(R_{1z}^2-R_{1x}^2)}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}\left(v_{i,k+1}-v_{i-1,k}\right)-\mu_{i+1/2\,,k-1/2}\left(v_{i+1,k}-v_{i,k-1}\right)\right]$$

(iv)

$$\frac{(R_{1z}^2+R_{1x}^2)}{2}\frac{\partial}{\partial z'}\left[\mu\frac{\partial v}{\partial z'}\right]\Rightarrow_{RF6}$$
$$\frac{(R_{1z}^2+R_{1x}^2)}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}\left(v_{i-1,k+1}-v_{i,k}\right)-\mu_{i+1/2\,,k-1/2}\left(v_{i,k}-v_{i+1,k-1}\right)\right]$$

3v.

$$-i\omega p_y\frac{R_{1x}}{\sqrt{2}}\left\{\left(\frac{\partial}{\partial x'}[\mu u]-\frac{\partial}{\partial z'}[\mu u]\right)+\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right\}\Rightarrow_{RF7-RF8+RF1-RF2}$$
$$-i\omega p_y\frac{R_{1x}}{4\Delta}\left[\mu_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}+u_{i,k}\right)-\mu_{i-1/2\,,k-1/2}\left(u_{i,k}+u_{i-1,k-1}\right)\right]$$
$$+i\omega p_y\frac{R_{1x}}{4\Delta}\left[\mu_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}+u_{i,k}\right)-\mu_{i+1/2\,,k-1/2}\left(u_{i,k}+u_{i+1,k-1}\right)\right]$$
$$-i\omega p_y\frac{R_{1x}}{4\Delta}\left[\lambda_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}-u_{i,k}\right)+\lambda_{i-1/2\,,k-1/2}\left(u_{i,k}-u_{i-1,k-1}\right)\right]$$
$$+i\omega p_y\frac{R_{1x}}{4\Delta}\left[\lambda_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}-u_{i,k}\right)+\lambda_{i+1/2\,,k-1/2}\left(u_{i,k}-u_{i+1,k-1}\right)\right]$$

4v.

$$
\begin{aligned}
&- i\omega p_y \frac{R_{1z}}{\sqrt{2}} \left\{ \left(\frac{\partial}{\partial z'}[\mu w] + \frac{\partial}{\partial x'}[\mu w] \right) + \lambda \left(\frac{\partial w}{\partial z'} + \frac{\partial w}{\partial x'} \right) \right\} \Rightarrow_{RF8+RF7+RF2+RF1} \\
&\quad - i\omega p_y \frac{R_{1z}}{4\Delta} \left[\mu_{i-1/2\,,k+1/2} \left(w_{i-1,k+1} + w_{i,k} \right) - \mu_{i+1/2\,,k-1/2} \left(w_{i,k} + w_{i+1,k-1} \right) \right] \\
&\quad - i\omega p_y \frac{R_{1z}}{4\Delta} \left[\mu_{i+1/2\,,k+1/2} \left(w_{i+1,k+1} + w_{i,k} \right) - \mu_{i-1/2\,,k-1/2} \left(w_{i,k} + w_{i-1,k-1} \right) \right] \\
&\quad - i\omega p_y \frac{R_{1z}}{4\Delta} \left[\lambda_{i+1/2\,,k+1/2} \left(w_{i+1,k+1} - w_{i,k} \right) + \lambda_{i-1/2\,,k-1/2} \left(w_{i,k} - w_{i-1,k-1} \right) \right] \\
&\quad\quad - i\omega p_y \frac{R_{1z}}{4\Delta} \left[\lambda_{i-1/2\,,k+1/2} \left(w_{i-1,k+1} - w_{i,k} \right) + \lambda_{i+1/2\,,k-1/2} \left(w_{i,k} - w_{i+1,k-1} \right) \right]
\end{aligned}
$$

Then outside of the PML we have the non rotated

$$\frac{A5VV}{v_{i,k}} = b\omega^2\rho'' - \frac{1}{\Delta^2}[\mu_{i\,,k+1/2} + \mu_{i\,,k-1/2} + \mu_{i+1/2\,,k} + \mu_{i-1/2\,,k}] \tag{A.3a}$$

$$\frac{A6VV}{v_{i,k+1}} = \frac{(1-b)}{4}\omega^2\rho'' + \frac{1}{\Delta^2}\mu_{i\,,k+1/2} \tag{A.3b}$$

$$\frac{A4VV}{v_{i,k-1}} = \frac{(1-b)}{4}\omega^2\rho'' + \frac{1}{\Delta^2}\mu_{i\,,k-1/2} \tag{A.3c}$$

$$\frac{A8VV}{v_{i+1,k}} = \frac{(1-b)}{4}\omega^2\rho'' + \frac{1}{\Delta^2}\mu_{i+1/2\,,k} \tag{A.3d}$$

$$\frac{A2VV}{v_{i-1,k}} = \frac{(1-b)}{4}\omega^2\rho'' + \frac{1}{\Delta^2}\mu_{i-1/2\,,k} \tag{A.3e}$$

$$\frac{A5VW}{w_{i,k}} = \frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i\,,k+1/2} - (\lambda-\mu)_{i\,,k-1/2}\right] \tag{A.3f}$$

$$\frac{A6VW}{w_{i,k+1}} = -\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k+1/2} \tag{A.3g}$$

$$\frac{A4VW}{w_{i,k-1}} = \frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k-1/2} \tag{A.3h}$$

$$\frac{A5VU}{u_{i,k}} = \frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i+1/2\,,k} - (\lambda-\mu)_{i-1/2\,,k}\right] \tag{A.3i}$$

$$\frac{A8VU}{u_{i+1,k}} = -\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i+1/2\,,k} \tag{A.3j}$$

$$\frac{A2VU}{u_{i-1,k}} = \frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i-1/2\,,k} \tag{A.3k}$$

Then the rotated forms outside of the PML are:

$$\frac{A5VV}{v_{i,k}} = -\frac{1}{2\Delta^2}[\mu_{i+1/2\,,k+1/2} + \mu_{i-1/2\,,k-1/2} + \mu_{i-1/2\,,k+1/2} + \mu_{i+1/2\,,k-1/2}] \tag{A.4a}$$

$$\frac{A9VV}{v_{i+1,k+1}} = \frac{1}{2\Delta^2}\mu_{i+1/2\,,k+1/2} \tag{A.4b}$$

$$\frac{A1VV}{v_{i-1,k-1}} = \frac{1}{2\Delta^2}\mu_{i-1/2\,,k-1/2} \tag{A.4c}$$

$$\frac{A3VV}{v_{i-1,k+1}} = \frac{1}{2\Delta^2}\mu_{i-1/2\,,k+1/2} \tag{A.4d}$$

$$\frac{A7VV}{v_{i+1,k-1}} = \frac{1}{2\Delta^2}\mu_{i+1/2\,,k-1/2} \tag{A.4e}$$

$$\frac{A6VV}{v_{i,k+1}} = 0 \tag{A.4f}$$

$$\frac{A8VV}{v_{i+1,k}} = 0 \tag{A.4g}$$

$$\frac{A2VV}{v_{i-1,k}} = 0 \tag{A.4h}$$

$$\frac{A4VV}{v_{i,k-1}} = 0 \tag{A.4i}$$

$$\frac{A5VU}{u_{i,k}} \Rightarrow$$
$$-\frac{i\omega p_y}{4\Delta}\left[(\lambda-\mu)_{i-1/2\,,k-1/2} + (\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{A.4j}$$

$$\frac{A9VU}{u_{i+1,k+1}} \Rightarrow -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{A.4k}$$

$$\frac{A1VU}{u_{i-1,k-1}} \Rightarrow \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{A.4l}$$

$$\frac{A3VU}{u_{i-1,k+1}} \Rightarrow \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{A.4m}$$

$$\frac{A7VU}{u_{i+1,k-1}} \Rightarrow -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{A.4n}$$

$$\frac{A5VW}{w_{i,k}} \Rightarrow$$
$$\frac{i\omega p_y}{4\Delta}\left[(\lambda-\mu)_{i+1/2\,,k+1/2} + (\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{A.4o}$$

$$\frac{A9VW}{w_{i+1,k+1}} \Rightarrow -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{A.4p}$$

$$\frac{A1VW}{w_{i-1,k-1}} \Rightarrow \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{A.4q}$$

$$\frac{A3VW}{w_{i-1,k+1}} \Rightarrow -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{A.4r}$$

$$\frac{A7VW}{w_{i+1,k-1}} \Rightarrow \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{A.4s}$$

A.3 Finite Difference Forms in w

Recalling our continuous non rotated equation of motion in w given by (1.13c)

$$-(\mu p_y^2-\rho)\omega^2 w+R_{2z}(\lambda+2\mu)\frac{\partial w}{\partial z}+R_{1z}^2\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z}\right]+R_{1x}R_{1z}\left(\frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right]\right)$$
$$+R_{2x}\left(\mu\frac{\partial w}{\partial x}\right)+R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial w}{\partial x}\right]-R_{1z}i\omega p_y\left(\frac{\partial}{\partial z}\left[\lambda v\right]+\mu\frac{\partial v}{\partial z}\right)$$

then we find finite difference forms for the non rotated part:

1w.

$$(\rho-p_y^2\mu)\omega^2 w=\rho'\omega^2 w\Rightarrow\quad b\omega^2\rho' w_{i,k}+\omega^2\rho'\frac{(1-b)}{4}\left(w_{i+1,k}+w_{i-1,k}+w_{i,k+1}+w_{i,k-1}\right)$$

2w.

$$R_{1z}^2\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial w}{\partial z}\right]\Rightarrow_{F7}\quad\frac{R_{1z}^2}{2\Delta}\left[(\lambda+2\mu)_{i,k+1/2}\left(w_{i,k+1}-w_{i,k}\right)-(\lambda+2\mu)_{i,k-1/2}\left(w_{i,k}+w_{i,k-1}\right)\right]$$

3w.

$$R_{1x}R_{1z}\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial z}\right]\Rightarrow_{F2}\quad\frac{R_{1x}R_{1z}}{4\Delta^2}\left[\mu_{i+1,k}\left(u_{i+1,k+1}-u_{i+1,k-1}\right)-\mu_{i-1,k}\left(u_{i-1,k+1}-u_{i-1,k-1}\right)\right]$$

4w.

$$R_{1x}^2\frac{\partial}{\partial x}\left[\mu\frac{\partial w}{\partial x}\right]\Rightarrow_{F1}\quad\frac{R_{1x}^2}{\Delta^2}\left[\mu_{i+1/2\,,k}\left(w_{i+1,k}-w_{i,k}\right)-\mu_{i-1/2\,,k}\left(w_{i,k}-w_{i-1,k}\right)\right]$$

5w.

$$R_{1x}R_{1z}\frac{\partial}{\partial z}\left[\lambda\frac{\partial u}{\partial x}\right]\Rightarrow_{F4}\quad\frac{R_{1x}R_{1z}}{4\Delta^2}\left[\lambda_{i,k+1}\left(u_{i+1,k+1}-u_{i-1,k+1}\right)-\lambda_{i,k-1}\left(u_{i+1,k-1}-u_{i-1,k-1}\right)\right]$$

6w.

$$-R_{1z}i\omega p_y\left(\frac{\partial}{\partial z}[\lambda v]\right)\Rightarrow_{F8}\quad-R_{1z}\frac{i\omega p_y}{2\Delta}\left[\lambda_{i,k+1/2}(v_{i,k+1}+v_{i,k})-\lambda_{i,k-1/2}(v_{i,k}+v_{i,k-1})\right]$$

7w.

$$-R_{1z}i\omega p_y\left(\mu\frac{\partial v}{\partial z}\right)\Rightarrow_{F7}\quad -R_{1z}\frac{i\omega p_y}{2\Delta}\left[\mu_{i,k+1/2}\left(v_{i,k+1}-v_{i,k}\right)+\mu_{i,k-1/2}\left(v_{i,k}-v_{i,k-1}\right)\right]$$

New terms

8w.

$$R_{2x}\left[\mu\frac{\partial w}{\partial x}\right]\Rightarrow_{F6}\quad \frac{R_{2x}}{2\Delta}\left[\mu_{i+1/2\,,k}\left(w_{i+1,k}-w_{i,k}\right)+\mu_{i-1/2\,,k}\left(w_{i,k}-w_{i-1,k}\right)\right]$$

9w.

$$R_{2z}(\lambda+2\mu)\frac{\partial w}{\partial z}\Rightarrow_{F7}\quad \frac{R_{2z}}{2\Delta}\left[(\lambda+2\mu)_{i,k+1/2}\left(w_{i,k+1}-w_{i,k}\right)+(\lambda+2\mu)_{i,k-1/2}\left(w_{i,k}-w_{i,k-1}\right)\right]$$

Then recalling our rotated continuous equation of motion in w given by (1.15c)

$$\begin{aligned}(\mu p_y^2-\rho)\omega^2 w = {}& \frac{R_{2z}}{\sqrt{2}}(\lambda+2\mu)\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)+\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[(\lambda+2\mu)\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right]\\ &+\frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right]+\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right]\right\}\\ &+\frac{R_{2x}}{\sqrt{2}}\left\{\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right\}+\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right]\\ &-R_{1z}i\frac{\omega p_y}{\sqrt{2}}\left\{\mu\left(\frac{\partial v}{\partial z'}+\frac{\partial v}{\partial x'}\right)+\frac{\partial}{\partial z'}[\lambda v]+\frac{\partial}{\partial x'}[\lambda v]\right\}\end{aligned}$$

we find finite difference forms 1w.

$$\begin{aligned}&\frac{R_{2z}}{\sqrt{2}}\left[(\lambda+2\mu)\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right]\Rightarrow_{RF1+RF2}\\ &\frac{R_{2z}}{4\Delta}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\left(w_{i+1,k+1}-w_{i,k}\right)+(\lambda+2\mu)_{i-1/2\,,k-1/2}\left(w_{i,k}-w_{i-1,k-1}\right)\right]\\ +&\frac{R_{2z}}{4\Delta}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2}\left(w_{i-1,k+1}-w_{i,k}\right)+(\lambda+2\mu)_{i+1/2\,,k-1/2}\left(w_{i,k}-w_{i+1,k-1}\right)\right]\end{aligned}$$

2w.

$$\frac{R_{1z}^2}{2}\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[(\lambda+2\mu)\left(\frac{\partial w}{\partial z'}+\frac{\partial w}{\partial x'}\right)\right]$$
$$=\frac{R_{1z}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial w}{\partial x'}\right]+\frac{R_{1z}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial w}{\partial z'}\right]+\frac{R_{1z}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial w}{\partial x'}\right]$$
$$+\frac{R_{1z}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial w}{\partial z'}\right]$$

(i)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial w}{\partial x'}\right]\Rightarrow_{RF3}$$
$$\frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\left(w_{i+1,k+1}-w_{i,k}\right)-(\lambda+2\mu)_{i-1/2\,,k-1/2}\left(w_{i,k}-w_{i-1,k-1}\right)\right]$$

(ii)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial x'}\left[(\lambda+2\mu)\frac{\partial w}{\partial z'}\right]\Rightarrow_{RF4}$$
$$\frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i+1/2\,,k+1/2}\left(w_{i,k+1}-w_{i+1,k}\right)-(\lambda+2\mu)_{i-1/2\,,k-1/2}\left(w_{i-1,k}-w_{i,k-1}\right)\right]$$

(iii)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial w}{\partial x'}\right]\Rightarrow_{RF5}$$
$$\frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2}\left(w_{i,k+1}-w_{i-1,k}\right)-(\lambda+2\mu)_{i+1/2\,,k-1/2}\left(w_{i+1,k}-w_{i,k-1}\right)\right]$$

(iv)

$$\frac{R_{1z}^2}{2}\frac{\partial}{\partial z'}\left[(\lambda+2\mu)\frac{\partial w}{\partial z'}\right]\Rightarrow_{RF6}$$
$$\frac{R_{1z}^2}{4\Delta^2}\left[(\lambda+2\mu)_{i-1/2\,,k+1/2}\left(w_{i-1,k+1}-w_{i,k}\right)-(\lambda+2\mu)_{i+1/2\,,k-1/2}\left(w_{i,k}-w_{i+1,k-1}\right)\right]$$

3w.

$$\frac{R_{1x}R_{1z}}{2}\left\{\left(\frac{\partial}{\partial z'}+\frac{\partial}{\partial x'}\right)\left[\lambda\left(\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}\right)\right]+\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial u}{\partial z'}+\frac{\partial u}{\partial x'}\right)\right]\right\}$$
$$+\frac{R_{1x}R_{1z}}{2}\left(-\frac{\partial}{\partial x'}\left[\lambda\frac{\partial u}{\partial z'}\right]+\frac{\partial}{\partial x'}\left[\lambda\frac{\partial u}{\partial x'}\right]-\frac{\partial}{\partial z'}\left[\lambda\frac{\partial u}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[\lambda\frac{\partial u}{\partial x'}\right]\right)$$
$$+\frac{R_{1x}R_{1z}}{2}\left(-\frac{\partial}{\partial z'}\left[\mu\frac{\partial u}{\partial x'}\right]+\frac{\partial}{\partial x'}\left[\mu\frac{\partial u}{\partial x'}\right]+\frac{\partial}{\partial x'}\left[\mu\frac{\partial u}{\partial z'}\right]-\frac{\partial}{\partial z'}\left[\mu\frac{\partial u}{\partial z'}\right]\right)$$
$$=\frac{R_{1x}R_{1z}}{2}\left(-\frac{\partial}{\partial x'}\left[(\lambda-\mu)\frac{\partial u}{\partial z'}\right]+\frac{\partial}{\partial x'}\left[(\lambda+\mu)\frac{\partial u}{\partial x'}\right]-\frac{\partial}{\partial z'}\left[(\lambda+\mu)\frac{\partial u}{\partial z'}\right]+\frac{\partial}{\partial z'}\left[(\lambda-\mu)\frac{\partial u}{\partial x'}\right]\right)$$

(i)

$$-\frac{\partial}{\partial x'}\left[(\lambda-\mu)\frac{\partial u}{\partial z'}\right]\Rightarrow_{RF4}$$
$$-\frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k+1/2}\left(u_{i,k+1}-u_{i+1,k}\right)-(\lambda-\mu)_{i-1/2\,,k-1/2}\left(u_{i-1,k}-u_{i,k-1}\right)\right]$$

(ii)

$$\frac{\partial}{\partial x'}\left[(\lambda+\mu)\frac{\partial u}{\partial x'}\right]\Rightarrow_{RF3}$$
$$\frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k+1/2}\left(u_{i+1,k+1}-u_{i,k}\right)-(\lambda+\mu)_{i-1/2\,,k-1/2}\left(u_{i,k}-u_{i-1,k-1}\right)\right]$$

(iii)

$$-\frac{\partial}{\partial z'}\left[(\lambda+\mu)\frac{\partial u}{\partial z'}\right]\Rightarrow_{RF6}$$
$$-\frac{R_{1x}R_{1z}}{2\Delta^2}\left[(\lambda+\mu)_{i-1/2\,,k+1/2}\left(u_{i-1,k+1}-u_{i,k}\right)-(\lambda+\mu)_{i+1/2\,,k-1/2}\left(u_{i,k}-u_{i+1,k-1}\right)\right]$$

(iv)

$$\frac{\partial}{\partial z'}\left[(\lambda-\mu)\frac{\partial u}{\partial x'}\right]\Rightarrow_{RF5}$$
$$\frac{R_{1x}R_{1z}}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k+1/2}\left(u_{i,k+1}-u_{i-1,k}\right)-(\lambda-\mu)_{i+1/2\,,k-1/2}\left(u_{i+1,k}-u_{i,k-1}\right)\right]$$

4w.

$$\frac{R_{2x}}{\sqrt{2}}\left[\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right]\Rightarrow_{RF1+RF2}$$
$$\frac{R_{2x}}{4\Delta}\left[\mu_{i+1/2\,,k+1/2}\left(w_{i+1,k+1}-w_{i,k}\right)+\mu_{i-1/2\,,k-1/2}\left(w_{i,k}-w_{i-1,k-1}\right)\right]$$
$$-\frac{R_{2x}}{4\Delta}\left[\mu_{i-1/2\,,k+1/2}\left(w_{i-1,k+1}-w_{i,k}\right)+\mu_{i+1/2\,,k-1/2}\left(w_{i,k}-w_{i+1,k-1}\right)\right]$$

5w.

$$\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial z'}\right)\left[\mu\left(\frac{\partial w}{\partial x'}-\frac{\partial w}{\partial z'}\right)\right]$$
$$=\frac{R_{1x}^2}{2}\left(\frac{\partial}{\partial z'}\left[\mu\frac{\partial w}{\partial z'}\right]-\frac{\partial}{\partial z'}\left[\mu\frac{\partial w}{\partial x'}\right]-\frac{\partial}{\partial x'}\left[\mu\frac{\partial w}{\partial z'}\right]+\frac{\partial}{\partial x'}\left[\mu\frac{\partial w}{\partial x'}\right]\right)$$

(i)

$$\frac{R_{1x}^2}{2}\frac{\partial}{\partial z'}\left[\mu\frac{\partial w}{\partial z'}\right]\Rightarrow_{RF6}\quad\frac{R_{1x}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}\left(w_{i-1,k+1}-w_{i,k}\right)-\mu_{i+1/2\,,k-1/2}\left(w_{i,k}-w_{i+1,k-1}\right)\right]$$

(ii)

$$-\frac{R_{1x}^2}{2}\frac{\partial}{\partial z'}\left[\mu\frac{\partial w}{\partial x'}\right]\Rightarrow_{RF5}\quad-\frac{R_{1x}^2}{4\Delta^2}\left[\mu_{i-1/2\,,k+1/2}\left(w_{i,k+1}-w_{i-1,k}\right)-\mu_{i+1/2\,,k-1/2}\left(w_{i+1,k}-w_{i,k-1}\right)\right]$$

(iii)

$$-\frac{R_{1x}^2}{2}\frac{\partial}{\partial x'}\left[\mu\frac{\partial w}{\partial z'}\right]\Rightarrow_{RF4}\quad-\frac{R_{1x}^2}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2}\left(w_{i,k+1}-w_{i+1,k}\right)-\mu_{i-1/2\,,k-1/2}\left(w_{i-1,k}-w_{i,k-1}\right)\right]$$

(iv)

$$\frac{R_{1x}^2}{2}\frac{\partial}{\partial x'}\left[\mu\frac{\partial w}{\partial x'}\right]\Rightarrow_{RF3}\quad\frac{R_{1x}^2}{4\Delta^2}\left[\mu_{i+1/2\,,k+1/2}\left(w_{i+1,k+1}-w_{i,k}\right)-\mu_{i-1/2\,,k-1/2}\left(w_{i,k}-w_{i-1,k-1}\right)\right]$$

6w.

$$-\frac{R_{1z}i\omega p_y}{\sqrt{2}}\left(\mu\left(\frac{\partial v}{\partial z'}+\frac{\partial v}{\partial x'}\right)+\left(\frac{\partial}{\partial z'}[\lambda v]+\frac{\partial}{\partial x'}[\lambda v]\right)\right)\Rightarrow_{RF2+RF1+RF8+RF7}$$
$$-\frac{R_{1z}i\omega p_y}{4\Delta}\left[\lambda_{i+1/2\,,k+1/2}\left(v_{i+1,k+1}+v_{i,k}\right)-\lambda_{i-1/2\,,k-1/2}\left(v_{i,k}+v_{i-1,k-1}\right)\right]$$
$$-\frac{R_{1z}i\omega p_y}{4\Delta}\left[\lambda_{i-1/2\,,k+1/2}\left(v_{i-1,k+1}+v_{i,k}\right)-\lambda_{i+1/2\,,k-1/2}\left(v_{i,k}+v_{i+1,k-1}\right)\right]$$
$$-\frac{R_{1z}i\omega p_y}{4\Delta}\left[\mu_{i+1/2\,,k+1/2}\left(v_{i+1,k+1}-v_{i,k}\right)+\mu_{i-1/2\,,k-1/2}\left(v_{i,k}-v_{i-1,k-1}\right)\right]$$
$$-\frac{R_{1z}i\omega p_y}{4\Delta}\left[\mu_{i-1/2\,,k+1/2}\left(v_{i-1,k+1}-v_{i,k}\right)+\mu_{i+1/2\,,k-1/2}\left(v_{i,k}-v_{i+1,k-1}\right)\right]$$

Then non-rotated equations of motion outside of the PML reduce to

$$\frac{A5WW}{w_{i,k}}=b\omega^2\rho'-\frac{1}{\Delta^2}\left[(\lambda+2\mu)_{i\,,k+1/2}\;+(\lambda+2\mu)_{i,\,,k-1/2}+\mu_{i+1/2\,,k}\;+\mu_{i-1/2\,,k}\right] \tag{A.5a}$$

$$\frac{A6WW}{w_{i,k+1}}=\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}(\lambda+2\mu)_{i\,,k+1/2} \tag{A.5b}$$

$$\frac{A4WW}{w_{i,k-1}}=\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}(\lambda+2\mu)_{i\,,k-1/2} \tag{A.5c}$$

$$\frac{A8WW}{w_{i+1,k}}=\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}\mu_{i+1/2\,,k} \tag{A.5d}$$

$$\frac{A2WW}{w_{i-1,k}}=\frac{(1-b)}{4}\omega^2\rho'+\frac{1}{\Delta^2}\mu_{i-1/2\,,k} \tag{A.5e}$$

$$\frac{A9WU}{u_{i+1\,,k+1}}=\frac{1}{4\Delta^2}(\lambda_{i\,,k+1}\;+\mu_{i+1\,,k}) \tag{A.5f}$$

$$\frac{A7WU}{u_{i+1\,,k-1}}=-\frac{1}{4\Delta^2}(\lambda_{i\,,k-1}\;+\mu_{i+1\,,k}) \tag{A.5g}$$

$$\frac{A3WU}{u_{i-1\,,k+1}}=-\frac{1}{4\Delta^2}(\lambda_{i\,,k+1}\;+\mu_{i-1\,,k}) \tag{A.5h}$$

$$\frac{A1WU}{u_{i-1\,,k-1}}=\frac{1}{4\Delta^2}(\lambda_{i\,,k-1}\;+\mu_{i-1\,,k}) \tag{A.5i}$$

$$\frac{A5WV}{v_{i,k}}=\frac{i\omega p_y}{2\Delta}\left[(\lambda-\mu)_{i\,,k-1/2}\;-(\lambda-\mu)_{i\,,k+1/2}\right] \tag{A.5j}$$

$$\frac{A6WV}{v_{i,k+1}}=-\frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k+1/2} \tag{A.5k}$$

$$\frac{A4WV}{v_{i,k-1}} = \frac{i\omega p_y}{2\Delta}(\lambda+\mu)_{i\,,k-1/2} \tag{A.5l}$$

And for the rotated forms outside of the PML:

$$\frac{A5WW}{w_{i,k}} = -\frac{1}{4\Delta^2}$$
$$\left[(\lambda+3\mu)_{i-1/2\,,k-1/2} + (\lambda+3\mu)_{i+1/2\,,k+1/2} + (\lambda+3\mu)_{i-1/2\,,k+1/2} + (\lambda+3\mu)_{i+1/2\,,k-1/2}\right] \tag{A.6a}$$

$$\frac{A9WW}{w_{i+1,k+1}} = \frac{1}{4\Delta^2}(\lambda+3\mu)_{i+1/2\,,k+1/2} \tag{A.6b}$$

$$\frac{A1WW}{w_{i-1,k-1}} = \frac{1}{4\Delta^2}(\lambda+3\mu)_{i-1/2\,,k-1/2} \tag{A.6c}$$

$$\frac{A3WW}{w_{i-1,k+1}} = \frac{1}{4\Delta^2}(\lambda+3\mu)_{i-1/2\,,k+1/2} \tag{A.6d}$$

$$\frac{A7WW}{w_{i+1,k-1}} = \frac{1}{4\Delta^2}(\lambda+3\mu)_{i+1/2\,,k-1/2} \tag{A.6e}$$

$$\frac{A6WW}{w_{i,k+1}} = \frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k+1/2} + (\lambda+\mu)_{i-1/2\,,k+1/2}\right] \tag{A.6f}$$

$$\frac{A8WW}{w_{i+1,k}} = -\frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k+1/2} + (\lambda+\mu)_{i+1/2\,,k-1/2}\right] \tag{A.6g}$$

$$\frac{A2WW}{w_{i-1,k}} = -\frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i-1/2\,,k-1/2} + (\lambda+\mu)_{i-1/2\,,k+1/2}\right] \tag{A.6h}$$

$$\frac{A4WW}{w_{i,k-1}} = \frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i+1/2\,,k-1/2} + (\lambda+\mu)_{i-1/2\,,k-1/2}\right] \tag{A.6i}$$

$$\frac{A5WU}{u_{i,k}}$$
$$= \frac{1}{4\Delta^2}\left[(\lambda+\mu)_{i-1/2\,,k+1/2} + (\lambda+\mu)_{i+1/2\,,k-1/2} - (\lambda+\mu)_{i+1/2\,,k+1/2} - (\lambda+\mu)_{i-1/2\,,k-1/2}\right] \tag{A.6j}$$

$$\frac{A8WU}{u_{i+1,k}} = \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k-1/2}\right] \tag{A.6k}$$

$$\frac{A6WU}{u_{i,k+1}} = \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k+1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2}\right] \tag{A.6l}$$

$$\frac{A2WU}{u_{i-1,k}} = \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i-1/2\,,k+1/2}\right] \tag{A.6m}$$

$$\frac{A4WU}{u_{i,k-1}} = \frac{1}{4\Delta^2}\left[(\lambda-\mu)_{i+1/2\,,k-1/2} - (\lambda-\mu)_{i-1/2\,,k-1/2}\right] \tag{A.6n}$$

$$\frac{A9WU}{u_{i+1,k+1}} = \frac{1}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{A.6o}$$

$$\frac{A1WU}{u_{i-1,k-1}} = \frac{1}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{A.6p}$$

$$\frac{A3WU}{u_{i-1,k+1}} = -\frac{1}{4\Delta^2}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{A.6q}$$

$$\frac{A7WU}{u_{i+1,k-1}} = -\frac{1}{4\Delta^2}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{A.6r}$$

$$\frac{A5WV}{v_{i,k}}$$
$$= \frac{i\omega p_y}{4\Delta}\left[(\lambda-\mu)_{i+1/2\,,k-1/2} + (\lambda-\mu)_{i-1/2\,,k-1/2} - (\lambda-\mu)_{i+1/2\,,k+1/2} - (\lambda-\mu)_{i-1/2\,,k+1/2}\right] \tag{A.6s}$$

$$\frac{A9WV}{v_{i+1,k+1}} = -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k+1/2} \tag{A.6t}$$

$$\frac{A1WV}{v_{i-1,k-1}} = \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k-1/2} \tag{A.6u}$$

$$\frac{A3WV}{v_{i-1,k+1}} = -\frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i-1/2\,,k+1/2} \tag{A.6v}$$

$$\frac{A7WV}{v_{i+1,k-1}} = \frac{i\omega p_y}{4\Delta}(\lambda+\mu)_{i+1/2\,,k-1/2} \tag{A.6w}$$